INTRODUCTION TO THE SCIENCE

OF

DYNAMICS

BY

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PHYSICAL LAWS.

TABLES OF MEASUREMENTS.

Useful and Important Numbers.

$\pi = 3{\cdot}14159265$, $\log \pi = 0{\cdot}4971499$, 1 radian $= 180^\circ/\pi = 57{\cdot}3^\circ$.

Mean value of $g = 980{\cdot}5$ tachs per sec., or $32\frac{1}{6}$ vels per sec.

Zero of the centigrade scale $= 273^\circ$ air thermometer scale, and $0^\circ A = -273^\circ C$.

Mean sea-level atmospheric pressure $= 76$ cm. of mercury at 0° in the latitude of Paris $= 14{\cdot}7$ lbs.-wt. per sq. in. $= 10\frac{1}{3}$ tonnes-wt. per sq. metre $= 1{\cdot}014$ megabarad.

Earth's mean radius $= 6470{\cdot}9$ kilometres $= 3958{\cdot}7$ miles.

Earth's mean density $= 5{\cdot}67$, and mass $= 6{\cdot}14 \times 10^{21}$ tonnes.

PREFACE.

The present text-book embraces Part I and the half of Part II of the author's Introduction to the Science of Dynamics, first printed in 1886, and contains as much of that work, as experience has shewn he is able to give to the two divisions of his pass class at the University, at the present stage of university education in Ontario. The present edition will, I trust, be found to be a great improvement on the last. It is, however, impossible to escape all errors, and any suggestions or corrections from students will be thankfully received.

The names *tach*, *gramtach*, and *dyntach* have been retained for the C.G.S. units of speed, momentum, and activity, as no other names as good as these have yet been proposed. The Canadian ton of 2.000 lbs. has been used in preference to the awkward English ton. Surely to call 112 lbs. a hundred-weight is unworthy of a scientific nation. Let such absurdities disappear, like that foolish but fast fading notion, that a knowledge of the dead languages is necessary to a liberal education, or that equally absurd one, that a knowledge of Hebrew should form an essential part of the education of a modern preacher.

It is difficult and I think pedantic for an author to attempt to enumerate the books and authors to whom he is indebted, but I cannot refrain from at least thankfully acknowledging my gratitude to my old teacher and friend, Prof. Tait, of Edinburgh University, to whose clear exposition of the great fundamental facts and laws underlying the constitution of the universe, so many thousands of students are indebted; and also my indebtedness to my

friend and former colleague, Prof. R. H. Smith, of London, emeritus professor of engineering in Mason College, Birmingham, for his trenchant criticism of the methods of dealing with some of the difficult problems in that only sure foundation of the higher problems in all the sciences, the science of Dynamics.

The full Table of Contents, as well as the lists of the Tables of Measurement and Physical Laws expounded in the text, which precede this preface, will, I trust, make reference to the text sufficiently easy to the student.

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Elmhurst, Kingston, Ont.
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INTRODUCTION.

All our knowledge of the material world is derived from *experience*, which can be conveniently divided into *observation* and *experiment*. Astronomy is an example of a science in which all our knowledge is primarily derived from simple observation, whereas in the science of electricity all important advances have been made by the performance of experiments. Hence, whilst the history of astronomy stretches over more than two thousand years, that of electricity hardly extends over two hundred.

Observation consists in simply observing with the aid only of our senses what is taking place in the material world.

Experiment is the controlling to a greater or less extent what is to take place, in order to find out what will take place under special circumstances.

What we observe and experiment with is *matter*. This term, like the terms *space*, *direction*, and *time*, it is impossible to define satisfactorily.

Space is limitless extension in all directions. It is the abode of matter, in which all motions take place, though itself immaterial. The term *matter* is applied to anything which is perceived by our senses, and which occupies space. A shadow can be perceived but is not matter, since it does not occupy space. So with motion, perplexity, anger, joy. The *Torricellian vacuum* occupies space, but it is not matter, since (as yet) it cannot be perceived by the senses.

All great advances in Science have been made by *measuring* what is observed. *Mathematics* may be defined as the science of measurement. It is divided generally into (1) Pure Mathematics, and (2) Applied Mathematics. In the former, measurements of space and time are principally considered. In the latter, besides space and time, the properties and conditions of matter, such as mass, weight, energy, temperature, potential, are measured. In a wider sense Applied Mathematics is known as *Natural Philosophy* or *Physics*. Natural Philosophy is the science which investigates and measures the properties of matter as discovered by direct observation and experiment and deduces the laws connecting these properties. So extensive, however, has our knowledge of the properties and conditions of matter become, that different branches of Natural Philosophy are conveniently separated from the parent stem. Chemistry, Astronomy, Geology, Biology, &c., though originally branches of Natural Philosophy, have put forth roots like the branches of the banyan tree and become themselves trees of knowledge, sending forth their own branches, and these in their turn new roots. But the same vital force permeates trunk and branches alike, and it is this vital force, under its new name *energy*, which now forms the subject-matter of physical science. Natural Philosophy or Physics is thus the science of energy, and is divided into the following principal parts: (1) Dynamics, which treats of *molar* energy, (2) Sound, (3) Heat, (4) Magnetism and Electricity, (5) Light and other kinds of *radiant* energy.

Before any measurements can be made, certain *units of measurement* must be fixed upon. Thus, the navigator measures the run of his ship in knots, the surveyor his land in acres, and states of heat are measured in thermometric degrees. Now, not only in different countries, but even in the same country, different units, bearing no simple relations to one another, are constantly used in

measurements of the same kind. In order to avoid all unnecessary calculations in the comparison of different observations, scientific men have agreed to adopt a uniform system of units. This is founded on the French system of units and is known as the *Centimetre-Gram-Second* or C. G. S. system. With the English *foot, pound,* and *second* as fundamental units an English system of units is formed called the F. P. S. system. In the following pages the student is exercised in the use of the C. G. S as well as the English units.

When for special measurements it is desirable to use larger or smaller units than the standards, these are formed in the C. G. S. system quite uniformly, except in measurements of *time*, by prefixing the words *deca, hecto, kilo, mega,* to the name of the standard to indicate multiples of 10, 10^2, 10^3, 10^6, times the standard unit, and by prefixing *deci, centi, milli,* to indicate submultiples of 10^{-1}, 10^{-2}, 10^{-3}. Taken in connection with the decimal notation in the writing of numbers, such a system of forming the multiples and submultiples saves all unnecessary calculations in reducing to the standard unit. In the English system of forming multiples and submultiples, the numbers seem to have been chosen with a view of containing as many prime factors as possible, an imaginary advantage which has occasioned a very great amount of unnecessary calculation. In conformity with the C. G. S. system of units, all temperatures in the following pages are given in degrees *centigrade.*

Units of Length.

10^3 millimetres = 10^2 centimetres = 10 decimetres = 1 metre = 10^{-1} decametre = 10^{-2} hectometre = 10^{-3} kilometre = 10^{-6} megametre.

3 feet = 1 yard, 6 feet = 1 fathom, 100 links = 1 chain = 22 yards, 5280 feet = 1760 yards = 80 chains = 1 mile.

Units of Surface.

1 are $=1$ square decametre $=10^2$ square metres $=10^6$ square centimetres.

1 acre $=10$ square chains, 640 acres $=1$ square mile.

Units of Volume.

1 litre $=1$ cubic decimetre $=10^3$ cubic centimetres.

1 gallon $=277{\cdot}274$ cubic inches, and holds 10 lbs. avoir. of water at 62° F.

Units of Mass.

1000 milligrams $=$ 100 centigrams $=$ 10 decigrams $=$ 1 gram $=10^{-1}$ decagram $=10^{-2}$ hectogram $=10^{-3}$ kilogram $=10^{-6}$ tonne.

1 pound (avoirdupois) $=$ 7000 grains, 1 English ton $=$ 2240 lbs., 1 Canadian ton $=$ 2000 lbs.

The following useful formulæ should be quite familiar to the student:

$\pi = 3\frac{1}{7}$, $3{\cdot}1416$, $355/113$, $3{\cdot}1415926536$.

Circumference of a circle $=\pi d = 2\pi r$

Area of a circle $=\pi r^2$

Surface of a sphere $=\pi d^2 = 4\pi r^2$

Volume of a sphere $=\frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$

INTRODUCTION TO THE SCIENCE OF DYNAMICS.

CHAPTER I.

Extension. Direction.

1. The student of elementary dynamics is not concerned with the ultimate structure of matter, of which various theories have been advanced by scientific men, but only with its properties. The principal of these which we shall consider are extension, inertia, mass, weight, and energy.

2. Any portion of matter is called a *body*. The grains of sand on the sea-shore, our own bodies, houses, the whole earth, the planets, the fixed stars, are examples of bodies. The expression of the fact that two or more bodies cannot at the same time occupy the same portion of space is known as the principle of *impenetrability*.

3. *Extension* is that property of matter implied in the statement that every body occupies a limited portion of space. Every body has therefore *form* or *shape*. The *volume* of a body is the measure of its extension. The term *bulk* is often used in the same sense. The internal volume of a body, *e.g.* that of a cup or of a hollow sphere, is the amount of space enclosed by the body, and is called its *capacity*.

4. Before any measurements can be made it is necessary to fix upon *units* or definite quantities of what we desire to measure, in terms of which all other quantities of the same kind are expressed by means of numbers. In measurements of extension three units are used, viz., units of length, area, and volume. Of these the unit of length

may be taken as a fundamental or independent unit, and the others made to depend upon it, and these are hence called derived units. In any system of units a *fundamental unit* is one whose magnitude is independent of that of any other unit, otherwise than as a mere multiple or submultiple of a unit of the same kind. A *derived unit* is one whose magnitude depends upon the magnitudes of one or more other units, but is not a mere multiple or submultiple of a unit of the same kind.

5. The English standard unit of length (or distance) is the yard, which is defined by Act of Parliament as the distance between two points on a bar of metal at a definite temperature. The French unit, the *metre*, although derived originally from the dimensions of the earth, is similarly defined. The unit of length adopted in the C. G. S. system of units is one of the submultiples of the French unit, viz., the *centimetre*, and its multiples and submultiples are the same as the French.

6. Whatever unit of length be used, it is found most convenient in measurements of surface to take as the unit of area (or surface) the area of a square of which the side is unit of length, or a multiple or submultiple thereof. Hence the C. G. S. unit of area is a *square centimetre*. The French unit of area, the *are*, is a square decametre.

7. Similarly the unit of volume is immediately and most conveniently derived from the unit of length by defining it as the volume of a cube of which the edge is unit of length or a multiple or submultiple of the unit of length. Hence the C. G. S. unit of volume is a *cubic centimetre*. The French unit of capacity, the *litre*, is a cubic decimetre, and the unit of volume, the *stere*, a cubic metre.

8. *Direction* is relative position irrespective of distance. It is the only property or characteristic of an indefinite straight line. It is the characteristic property of

motion (Chap. II), and thus indicates how one must go from one point of space to reach another. An *angle* is difference of direction. The unit of angle in common measurements is the *degree*, which is the 90th part of a right angle. It cannot be said that the unit of angle is derivable from the unit of length, but it is most conveniently measured as the ratio of two lengths. This will be understood when we remember that if a circle be described with the vertex of an angle as centre and with any radius, the magnitude of the angle is measured by the ratio of the length of the arc on which it stands to the length of the radius. We may, therefore, define the unit angle as that angle which is subtended by an arc of unit length at the centre of a circle of unit radius. This is just the same as the angle which is subtended by an arc, whose length is equal to the radius, at the centre of any circle whatsoever, and is called a *radian*.

9. As mentioned above, we learn from elementary geometry that, whatever unit of angle be adopted, the following formula expresses the relation between the length of any arc (a) of a circle, the length of the radius (r), and the magnitude of the angle (i) subtended at the centre of the circle by the arc, $a = C\,ri$, where C is a constant number, whose value depends upon the unit of angle adopted. If we measure (i) in radians, this reduces to the simple form

$$a = ri$$

10. *To express the value of a radian in degrees.*

Since the arc subtended by a straight angle, *i.e.* by two right angles $= \pi r$, if (i) be the measure of two right angles in radians, we get $\pi r = ri$, $\therefore\ i = \pi$, *i.e.* two right angles $= \pi$ radians, and $\therefore$ a radian $= 180^{0}/\pi = 57^{0}\ 17'\ 44''\!\cdot\!8$ true to the tenth part of a second of angle, or very nearly $57^{0}\!\cdot\!3$.

11. In many dynamical investigations it is unnecessary to consider in any way the dimensions of a body, or the

distances between the different parts of a system of bodies. When this is the case, the body or system of bodies is called a *particle* or *material particle.* Thus in the explanation of the seasons, or of the phases of the moon, the earth or moon is a body, as we cannot neglect its dimensions, whereas in the determination of a planet's position in the sphere of the heavens at any time, the planet is a *particle.* In considering the proper motion of the solar system amongst the fixed stars, the sun, and indeed the whole solar system, are merely particles. The term particle is frequently defined as an indefinitely small body. The terms small and large are merely *relative*, and what is small at one time or from one point of view is large at another or from another point of view. In looking at a star through a large telescope we generally speak of the star as a particle, and of the telescope as a large body, and yet the star is immeasurably larger than the telescope. A grain of sand or a mote of salt we generally consider a very small body, but if it gets into one's eye, its size is enormous.

12. The most generally accepted theory of the ultimate structure of matter at the present day is known as the *atomic theory.* According to this theory matter is not infinitely divisible, but consists ultimately of excessively small indivisible particles. The smallest portion of any substance, beyond which mechanical sub-division is supposed to be impossible, is called a *molecule.* A molecule may, however, be *chemically* divided into *atoms.* Thus a molecule of water (H_2O) may be chemically divided into three atoms, two of Hydrogen, and one of Oxygen.

Examination I.

1. Define matter, and distinguish between the terms body, particle, molecule, atom.

2. Define extension, space, length, and area.

3. What is impenetrability? Give illustrations of the apparent contradiction of this principle, and explain them.

4. Distinguish between volume, bulk, and capacity.

5. What is a unit of measurement? Give the C. G. S. and F. P. S. units of length, area, and volume.

6. What is a metre, a litre, a stere? State the numerical relations between the are and sq. cm., and between the litre, stere, and cub. cm.

7. Define direction and angle. Name and define the principal units of angle.

8. Give in radians the angles of an equilateral triangle, a right angle, 30^0, a circumangle, and express a radian and $\frac{2}{3}\pi$ radians in degrees.

9. If the unit of length be a yard, and the unit of angle a right angle, what must be the value of C in the formula $a = C\ r\ i$? Is the value of C in this formula dependent on the unit of length? Why?

Exercise I.

The following examples in mensuration are appended to exercise the student in the use of the C. G. S. units and also of logarithmic tables to which he should early accustom himself. $Log\ \pi = 0{\cdot}4971499$.

1. The great pyramid of Gizeh is a regular pyramid on a square base. The original length of an edge of the base was 220·42m., and of a slant edge 232·865m.; find (1) the area of the ground on which it stands, (2) the exposed area of the pyramid, (3) the volume.

2. Assuming the earth to be a sphere, and that the length of an arc of a degree on a meridian is equal to 111·19 kilom., find (1) the length of the diameter, (2) the area of the earth's surface, (3) the volume.

3. If the nature of the earth's crust be known to a depth of 8 kilometres, find the ratio of the known to the unknown volume, supposing the earth to be a sphere of 6370·9 kilometres radius.

4. On the same supposition, how much of the earth's surface could a person see who was at a height of 4 kilometres above the sea level ?

5. If the atmosphere extend to a height of 70 kilometres, what is the ratio of its volume to that of the solid and liquid earth ?

6. Compare the earth's surface (taken as 100) with the torrid, temperate, and frigid zones of the earth, supposing the first to extend to an angular distance of $23°30'$ from the equator, and the last to a distance of $23°30'$ from each pole.

7. Two sectors of circles have equal areas, and the radii are as 1 to 2 ; find the ratio of the angles.

8. A gravel walk of uniform breadth is made round a rectangular grass-plot, the sides of which are 20 and 30 metres ; find the breadth of the walk, if its area be three-tenths of that of the grass-plot.

9. Find the number of litres of air in a room whose dimensions are 12·50 m., 5·45 m., and 3·70 m.

10. Find in radians the angle of a sector of a circle, the radius of which is 20 metres, and the area a deciare.

11. The horizontal parallax of the sun (*i.e.* the angle subtended by the earth's radius at the sun) is $8''·85$, and of the moon $57'\ 3''$; find the distances of these bodies in terms of the earth's radius.

12. Find also, in terms of a great circle of the earth, the areas of the moon's orbit and of the ecliptic, supposing these to be circles.

13. Find the circumference and area of the circle of latitude passing through Kingston, Ont., latitude 44° 13′, (See ex. 3).

14. A pendulum whose length is $1\frac{1}{2}$ metre swings through an arc whose chord is a decimetre; find the angle and the length of the arc of oscillation.

15. What must be the diameter and surface of a spherical balloon that its capacity may be 150000 litres?

ANSWERS.

When no unit is appended to an answer, the units of the C. G. S. or F. P. S. system are to be understood. When the answer cannot be expressed exactly by a number, the answer given is true to the last figure.

1. 485·850 ares; 904·313 ares; 2·80195 megasteres.
2. 12741·4 kilom.; $5{\cdot}10019 \times 10^{12}$ ares; $1{\cdot}08306 \times 10^{12}$ cub. kilom. 3. 1:265. 4. 160018 sq. kilom. 5. 1:30.
6. 100: 40: 52: 8. 7. 4:1. 8. 168·6. 9. 252062·5.
10. 1/20. 11. 23307; 60·3. 12. 3631; $5{\cdot}432 \times 10^{8}$.
13. 28689·5 kilom.; $6{\cdot}54991 \times 10^{7}$ sq. kilom.
14. 3° 49′ 13″·7; 10·002. 15. 659; 1·365 are.

CHAPTER II.

Motion. Velocity.

13. *Motion* is change of position. Although the ideas conveyed by the terms matter and motion are quite different, yet it is evident that all the motions we are cognizant of are the motions of matter directly, or are indirectly produced by motions of matter. Thus the motion we see when a boy throws a stone is the motion of the stone directly. A *wave*, on the other hand, which is *motion of form*, is not directly the motion of the medium through which the wave is passing, but is indirectly produced by the motion of this medium. What of the motion of a shadow, or of the sphere of the heavens?

14. The opposite (or the zero) of motion of *rest*. All the motion or rest of a body that we can know of is *relative*, *i.e.* with respect to some other body. In infinite space *absolute* motion or rest is indeterminable, if indeed conceivable. When we speak of rest and motion we generally mean either relatively to our own bodies, or relatively to our abode in space, the Earth. Every body is simultaneously at rest and in motion. When a person is sitting at ease in a railway carriage, he is said to be at rest. But this is merely *relatively* to the train. Relatively to the earth he is moving as fast as the train is, and when we consider that the earth is rotating about its axis, is further revolving around the sun, and with the sun and other members of the solar system careering through space, it is easily seen how complex is the person's motion. *The aim of the physicist is to determine those conditions of matter and motion which, apart from the world of sensation, thought, and consciousness, constitute the life of the universe.*

15. *Time* is continuous and limitless duration or existence, marking out the succession of events. In dynamical science it is conceived as a uniformly increasing quantity. It might also be defined as the immeasurable flow or continuity of instants, and is provisionally measured by the rotation of the sphere of the heavens.

16. *Velocity* is time-rate of motion, *i.e.* rate of change of position per unit of time. By *rate* is meant here degree of quickness. When two bodies are moving, and one moves over a greater distance in the same time than the other, the velocity of the former is said to be the greater. Velocity has *direction* as well as *magnitude*. The term *speed* is used for magnitude of velocity irrespective of direction. In any motion of a body the velocity may be *uniform*, *i.e.* the same throughout the motion, or it may be *variable*, *i.e.* continuously or at intervals changing during the motion. Similarly a speed may be *constant* or variable. The velocities of all bodies that we see moving are really variable. The motions of the hands of a chronometer, or the rotations about their axes of the different members of the solar system, are cases of motion in which the speeds are nearly constant. The test of constant speed is that equal distances are moved over in equal times, *however small these times may be.*

17. What, however, we naturally ask, are *equal times?* We look at our clocks or watches and say that they tell us equal times. Some watches go slow, others go fast, and how are we to know which go right? It is well-known that our clocks and watches are regulated by the apparent motion of the sun in the sphere of the heavens. This motion is the resultant of two motions, viz., (1) the apparent rotation of the sphere of the heavens, produced by the real rotation of the earth, which takes place in a *sidereal day;* and (2) the apparent revolution of the sun in the ecliptic in a *sidereal year*, produced by the real revolution of the

earth in its orbit around the sun. As the sidereal year is estimated in sidereal days, we find that ultimately the apparent rotation of the sphere of the heavens is our standard measurer of time, and we *define* equal times as times in which the sphere of the heavens apparently rotates through equal angles. Whether the term equality is rightly applied to such times or not is a legitimate enquiry.

The mean or average time in which the sun apparently rotates about the earth is called a *mean solar day*, and our unit of time, the *second*, is a well-known fraction of the mean solar day. In the C. G. S. and F. P. S. systems of units of measurement, the unit of time is, like the unit of length. one of the fundamental units.

18. Speed is measured by the number of units of length passed over in a unit of time. The unit of speed is derived immediately from the units of length and time ; it is the speed in which a unit of length is passed over in a unit of time. Hence the C. G. S. unit of speed is 1 centimetre per second. It will be convenient to call this a *tach*. The F. P. S. unit of speed is 1 foot per second, and is called a *vel*. If s be the distance in centimetres or feet described in t seconds by a body moving with a constant speed of v tachs or vels, then $s = vt$, and $v = s/t$.

19. When a body is moving with variable speed it has of course a definite speed at every instant, which is measured by the number of units of length which *would be* passed over in a unit of time, if for such a period from the instant in question the speed did not change. Hence we talk of a ship sailing at the rate of 12 knots an hour, or of a man walking at the rate of 4 miles an hour, although the speed of the ship or of the man may not be the same for any two consecutive seconds. When a body is moving with variable speed, the equation $v = s/t$ gives the mean speed during the time t, and, by taking t small enough,

we can approximate in any degree of exactitude to the speed at any instant.

20. A velocity can be completely represented by a straight line, the direction of the line representing the direction of the motion (the tangent to the path of the moving particle at the instant in question), and the length of the line representing the speed. Since a body may move in two directions along a line, the one being opposite to the other, it is convenient to distinguish these by the signs + and −, as is customary in the applications of algebra to geometry. If AB be a straight line, and a velocity in the direction of AB be called +, a velocity in the direction BA will be called −.

Angular Motion.

21. *Angular velocity* is rate of change of direction (of one point with respect to another) per unit of time.

EXAMPLE.—When a particle moves in a circle the time-rate of change of the angle, which the radius through the particle makes with a fixed radius, is the angular velocity of the particle about the centre of the circle.

If the angles described by the radius through the particle be equal in equal times, however small these may be, then the angular velocity is uniform and is measured by the angle described in a unit of time. The unit of angular velocity is that in which a unit of angle is described in a unit of time, *i.e.*, 1 radian per second.

When the angular velocity is variable, the angular velocity *at any instant* is measured by the angle which *would be* described in a unit of time, if for such a period from the instant in question the angular velocity did not change.

Corresponding to the equation $s = vt$ (Art. 18), we evidently have the equation $i = ot$, in which i is the angle described in time t with angular velocity o.

22. From the formula $a=ri$ (Art. 9) it follows at once that if v represent in tachs or vels the speed of a particle, moving in the circumference of a circle, r the radius in centimetres or feet, and o the angular velocity about the centre in radians per second,

$$v=ro, \text{ and } o=v/r.$$

23. If the angular velocity be *like* that of the hands of a watch, it is represented by the − sign, and if *unlike*, by the + sign. Let it be carefully observed, however, that the sign given to the angular velocity of a body depends upon the side of the plane of motion from which the motion is observed. Thus, if we could see the motion of the hands of a watch through the back of the watch, the angular velocity would be +. If we look northwards at the rotation of the sphere of the heavens it seems to be +, and if we look southwards it seems to be −.

Just as − linear motion is the *plane image* (*i.e.* the image in a plane mirror) of + motion, so − angular motion is the plane image of +, and vice versa.

Angular velocity is completely represented by a number with the sign + or − prefixed to it.

Examination II.

1. Define motion, rest, velocity, speed.

2. What is a wave? How is it produced? Give examples.

3. Illustrate the meanings of the terms *relative* and *absolute* with respect to extension, motion, and direction.

4. Distinguish between uniform and variable velocity, and define the speed of a body *at any instant* when the velocity is variable.

5. What is the test of constant speed?

6. Define time, equal times, and the unit of time.

7. Distinguish between fundamental and derived units, and give examples of each.

8. Name and define the C. G. S. and F. P. S. units of speed.

9. Give the relation between s, v, t in linear motion, and between θ, ω, t in angular motion.

10. Define a sidereal and a mean solar day, and give the numerical relations between them and a sidereal year.

11. How may velocity, speed, and angular velocity be completely represented ?

12. Define angular velocity, and the unit thereof.

13. Give and prove the relation between the speed and angular velocity about the centre of a circle of a body moving in the circumference.

14. Distinguish between + and − angular velocity. How are they related to one another ?

Exercise II.

1. A body has a speed of 10 tachs, how long will it take to pass over 600 metres ?

2. Which is greater a speed of 72 tachs or one of 252 metres per hour, and by how much ?

3. Express a speed of 72 kilometres per hour in decimetres per minute.

4. If a line a foot long represent a velocity of 3·75 miles per hour, what length of line would represent a velocity of 80 yards per minute ?

5. Two bodies start from the same point, the one 10 minutes after the other, and travel in perpendicular directions with speeds of 120 tachs and 100 metres per minute. How far apart will they be in an hour from the starting of the first ?

6. Two travellers leave the same place at the same time in directions inclined to one another at an angle of $\pi/3$,

and each travels with a speed of 166 tachs, how far apart will they be in two hours ?

7. A man two metres high walks in a straight line at the rate of 6 kilometres an hour away from a lighted lamp 3 metres high ; find in tachs the speed of the end of his shadow, and the rate at which his shadow lengthens.

8. If 3 minutes be the unit of time, and 50 decimetres the unit of length, what number measures the average rate of walking of a person who goes over 40 kilometres in 12 hours ?

9. If 7 metres per 3 minutes be the unit of speed, and 4 decimetres the unit of length, what must be the unit of time ?

10. If 3 metres per 7 minutes be the unit of speed, and 4 seconds the unit of time, what must be the unit of length ?

11. A body moving uniformly in a circle describes the circumference twice in 3 minutes, what is the measure of its angular velocity about the centre ?

12. What is the angular velocity of any body on the earth's surface due to the earth's rotation ?

13. The diameter of the driving wheel of a locomotive is 2 metres, what is the angular velocity of a point on the wheel about the centre, when the train is moving at the rate of 80 kilometres an hour ?

14. A body moving in the circumference of a circle of radius 10 has unit angular velocity about the centre ; find the space described in 10 seconds, and the time taken to complete a revolution.

ANSWERS.

1. 100 min. 2. 72 tachs ; 65 tachs. 3. 12000.
4. $\frac{8}{11}$ ft. 5. 660775. 6. 11952m. 7. 500; 333·$\dot{3}$.
8. 33·3. 9. 10 $\frac{2}{7}$ sec. 10. 2 $\frac{6}{7}$ cm. 11. $\pi/45$.
12. $\pi/43200$. 13. 22·2. 14. 1 m.; 2 π sec.

Chapter III.

Acceleration.

24. Just as a body's position may change, giving rise to motion, so a body's velocity may change, giving rise to *acceleration.*

Acceleration is change of velocity, not merely change of speed. A body's velocity may change in magnitude only, or in direction only, or in both magnitude and direction. The *total acceleration* during any time is the whole change of velocity during that time. The *acceleration at any instant* is the rate of change of velocity per unit of time at that instant.

The rate at which a body's velocity changes may be slow or fast. Compare the accelerations of trains in a long railway like the Canada Pacific, in which the stations are far apart, with the accelerations of trains in large cities, run for the convenience of passengers hurrying from one part of the city to another, such as on the Underground Railroad in London or on the Elevated Railroad in New York.

25. Acceleration has direction as well as magnitude, and may be uniform or variable. An acceleration is uniform when equal changes of *velocity* take place in equal times, however small these times may be, and is then measured by the velocity acquired in unit of time.

The direction of a body's acceleration may or may not be the same as the direction of its motion. We shall first consider acceleration, the direction of which is the same as that of the body's motion or opposite thereto. The effect of such an acceleration is evidently to change a body's speed without changing its direction of motion. If the direction of motion be considered +, then the acceleration will be + or − according as the speed is increasing or decreasing.

26. The *systematic* unit of acceleration (in magnitude) is unit of speed per unit of time. Hence the C. G. S. unit of acceleration is 1 tach per second, and the F. P. S. unit is 1 vel per second.

Observe carefully that the unit of acceleration, by involving the units of speed and time, involves the unit of length once and the unit of time twice. This must be particularly attended to if in the solution of a problem the units require to be changed. Thus a tach is represented by $\frac{60}{100}$ if a metre and minute be the units of length and time, but with the same units 1 tach per second will be represented by $\frac{60 \cdot 60}{100}$. One of the most important cases of the motion we are now considering is that of a body moving *vertically* upwards or downwards *in vacuo*. Such a body has a uniform acceleration *vertically downwards*. Its value, denoted by g, depends upon position, the mean value over the earth's surface, at the sea level, being 980·5 tachs per sec., or $32\frac{1}{6}$ vels per sec., nearly.

If a represent the acceleration of a body uniformly accelerated in the direction of its motion ($+ ly$ or $- ly$), and v denote the whole change of speed in time t, then

$$v = at, \text{ and } a = v/t.$$

27. A body may have a uniform acceleration which is different in direction from the direction of motion. The resultant motion of the body in this case is very different from that of the preceding case. Such would be the motion of a body near the earth's surface moving *in vacuo* in any but a vertical direction; it is very nearly that of a leaden bullet projected in the air in any but a vertical direction, and with a small speed. Such a body's speed will be always changing, though not at a constant rate, and the direction of motion will be always changing, so that the path described will be a parabola with its axis in the direction of acceleration. The parabolic path is well seen in the motion of a jet of water.

28. Again, a body may have an acceleration constant in magnitude but not in direction. Any body revolving uniformly in a circle (which is approximately the motion of the moon in its orbit) has such an acceleration. If o denote the angular velocity of the body about the centre, and r the radius of the circle, it can be shewn that the magnitude of acceleration is measured by ro^2, but the direction of acceleration is always towards the centre of the circle, and therefore changing at every instant.

29. When a body's acceleration is variable, the acceleration at any instant is measured by the number of units of velocity by which the body's velocity *would be* changed in a unit of time, if for such a period from the instant in question the acceleration remained uniform. When the acceleration is variable, the formula $a = v/t$ gives the average acceleration during the time t, and by taking t small enough we can approximate as closely as we please to the acceleration at the beginning of time t.

30. A bullet shot vertically upwards with great speed is an example of a body whose acceleration is constant in direction, but variable in magnitude on account of the varying resistance of the air. If the ball be shot in any but a vertical direction we have a case of motion in which the acceleration is always changing both in magnitude and direction.

31. Acceleration, like velocity, is completely represented by a straight line, the direction of the line being the direction of acceleration, and the length of the line representing the magnitude of the acceleration.

Examination III.

1. Define acceleration, total acceleration, and acceleration at any instant.

2. Define the unit of acceleration, What fundamental units does it involve ?

3. What is the acceleration of a falling body? What of a body rising upwards?

4. Under what conditions is the formula $v=at$ true? What is the test of uniform acceleration?

5. How is acceleration measured when variable?

6. Give examples of bodies having accelerations, (*a*) uniform; (*b*) variable, 1) in direction only, 2) in magnitude only, 3) in both direction and magnitude.

7. Shew that an acceleration of a metre per minute per second is equal to an acceleration of a metre per second per minute.

Exercise III.

In the following examples the acceleration is supposed to be uniform and in the direction of motion.

1. A body has an acceleration of 20 tachs per sec.; find in decimetres per minute the speed acquired in an hour.

2. Express the acceleration of a body falling in vacuo (980·5) in units of a metre and hour.

3. The acceleration due to the weight of a body is $32\frac{1}{6}$ vels per sec.; find the same in units of a yard and minute.

4. A body is thrown vertically upwards with a speed of 6000 tachs; what is its velocity at the end of 4 and of 8 seconds, neglecting the resistance of the air?

5. A body uniformly accelerated starts with a speed of 6 metres per minute, and in half an hour has a speed of 36 kilometres per hour; find the acceleration in tachs per second.

6. Compare the acceleration 2 tachs per sec. with that in which a speed of 1800 metres per hour is acquired in an hour.

7. Compare an acceleration 3 when a yard and minute are the fundamental units with an acceleration 1 when a foot and second are the fundamental units.

8. If 1 tach per 10 seconds were the unit of acceleration, what would be the measure of an acceleration of 10 tachs per second ?

9. If 6 kilometres per second per minute were the unit of acceleration, and 1 metre the unit of length, what would be the unit of time ?

10. If 1 decimetre per hour per second were the unit of acceleration, and 1 metre per minute the unit of speed, what would be the units of length and time in a scientific *system* of units ?

11. What is the difference between an acceleration of a metre per hour per second and one of a metre per minute per minute ?

12. Find in tachs per second the difference between an acceleration of 24 metres per minute per second and one of 21·6 kilometres per minute per hour.

13. If 216 kilometres per minute per hour be the unit of acceleration, and a second be the unit of time, what must be the unit of length ?

14. If the unit of speed be 96 metres per 15 minutes, and 10 seconds be the unit of time, express in C. G. S. measure the unit of acceleration.

15. If the unit of speed be 5 tachs, and 3 metres be the unit of length, express in kilometres per hour per hour the unit of acceleration.

16. If 7 metres be the unit of length, and 3 minutes the unit of time, what speed in tachs will a body acquire in half an hour with an acceleration 18 ?

17. A body starts with a speed of 120 tachs, and has an acceleration of 6 metres per minute per minute; another starts at the same time from rest with an acceleration of 36 kilometres per hour per hour; when will their speeds be equal ?

18. If 5 inches represent an acceleration of 10 tachs per minute, what length of line will represent an acceleration of 6 when a metre and minute are the units of length and time ?

19. A body is thrown vertically upwards with a speed of 10 kilotachs; after how many seconds will it be moving downwards with a speed of 5 kilotachs ?

20. The values of g were represented by two different nations by 12 and 25, and the speed of sound in air at 0° by numbers which were as 6 : 5; find the ratios of their units of length and time.

21. If g be represented by $1754\frac{6}{11}$ and an acre by 10 in a system of units; find what must be the units of speed and time.

Answers.

1. 432000. 2. 127072800. 3. 38600.
4. 2078 upwards; 1844 downwards. 5. $\frac{11}{20}$. 6. 144:1.
7. 1:400. 8. 100. 9. $\frac{1}{10}$ sec. 10. 10 m.; 10 min.
11. 0. 12. 30. 13. 1 m. 14. $1{\cdot}0\dot{6}$. 15. 10·8.
16. 700. 17. 18 min. 18. 5 inches. 19. 15·3.
20. 1 : 3; 2 : 5. 21. 1·1 vel; 1 minute.

CHAPTER IV.

Uniformly Accelerated Motion

32. We shall in this chapter consider more fully the nature of the motion of a body which is uniformly accelerated in the direction of its motion. If u be the velocity at any instant and a the acceleration, the velocity at the end of any time t will evidently be $u+at$; denoting this by v we get the first equation of motion,

$$v = u + at.$$

33. *To determine the space described in the time t.*

Since the velocity during the time t increases *uniformly* from u to $u+at$, the *average velocity* is $\frac{1}{2}\{u+(u+at)\}$ or $u+\frac{1}{2}at$. If a body moved *uniformly* during the time t with this speed, the space described would be $(u+\frac{1}{2}at)t$, or $ut+\frac{1}{2}at^2$. This will evidently be also the space described during the time t by a body whose velocity increases *uniformly* from u to $u+at$ in that time. Hence if a body has an initial velocity u, and an acceleration a in the direction of its motion, and if s denote the distance described in the time t,

$$s = ut + \tfrac{1}{2}at^2.$$

34. The following may be considered by the student a more rigorous proof of the same result:

Let the time t be divided into *any* large number of equal parts. If n denote the number, the duration of each little interval will be t/n. The velocities at the *beginning* of each of the little intervals will be

$$u\ ,\ u+a\frac{t}{n},\ u+2a\frac{t}{n},\ \dots\dots\dots\dots\dots\dots\ u+(n-1)a\frac{t}{n}.$$

The velocities at the *end* of each of the little intervals will be

$$u+a\frac{t}{n},\ u+2a\frac{t}{n},\ u+3a\frac{t}{n},\ \dots\dots\dots\dots\dots\ u+na\frac{t}{n}.$$

Suppose now that a body A moved *uniformly* during each little interval with the velocity indicated above at the beginning of each interval; if s_1 be the whole distance moved over during the n intervals. *i.e.* during the time t,

$$s_1 = \frac{t}{n}\left\{u + (u + a\frac{t}{n}) + (u + 2a\frac{t}{n}) + \ldots (u + \overline{n-1}a\frac{t}{n}\right\}$$

$$= ut + a\frac{t^2}{n^2} \cdot \left\{1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots \overline{n-1}\right\}$$

$$= ut + a\frac{t^2}{n^2} \cdot \frac{n(n-1)}{2} = ut + \tfrac{1}{2}at^2\left\{1 - \frac{1}{n}\right\}$$

Similarly if s_2 be the whole distance moved over by a body B, which moved uniformly during each little interval with the velocity indicated above at the end of each interval,

$$s_2 = ut + \tfrac{1}{2}at^2\left\{1 + \frac{1}{n}\right\}$$

Now if s be the whole space described during the time t by the body uniformly accelerated, it is evident that s is greater than s_1 and less than s_2, for the body A moved during each little interval with the *least* velocity, which the body uniformly accelerated had during that interval, and the body B with the *greatest*.

$$\therefore\ s > ut + \tfrac{1}{2}at^2\left\{1 - \frac{1}{n}\right\}$$

$$< ut + \tfrac{1}{2}at^2\left\{1 + \frac{1}{n}\right\}$$

Now these two quantities between which s lies differ only in the sign of $1/n$. What is n? n is *any number whatsoever*, and may be made as large as you please. But by taking n large enough, $1 - 1/n$ and $1 + 1/n$ may be made to differ from 1 by as small a fraction as you please. Hence when n becomes *indefinitely* great, the motions of A and B do not differ from the motion of the body uniformly accelerated, and the three quantities s, s_1, s_2 become $ut + \frac{1}{2}at^2$.

35. From the equations of uniformly accelerated motion just determined

$$v = u + at \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$s = ut + \tfrac{1}{2}at^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

we derive by algebraic analysis the following useful though not independent equation:

$$v^2 = u^2 + 2as \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Cor. 1. If the acceleration be opposite in direction to that of motion it must be represented by $-a$. and the equations become

$$v = u - at \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$
$$s = ut - \tfrac{1}{2}at^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$
$$v^2 = u^2 - 2as \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Cor. 2. If the body start from rest, $u = 0$ and the equations become

$$v = at \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$
$$s = \tfrac{1}{2}at^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$
$$v^2 = 2as \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Comparing (2) or (5) with (8) we might say that ut is the distance described in virtue of the speed u, and $\frac{1}{2}at^2$ that described in virtue of the acceleration a.

36. As already stated in art. 26, the motion of a body moving freely in a vertical direction is of the character we have been considering. Strictly speaking, this applies only to bodies moving *in vacuo*, but unless the velocity be great we may often neglect the action of the atmosphere.

Let us consider the motion of a body thrown vertically upwards with a velocity u.

1). *How long will it rise?*

It rises until its velocity is zero. Hence from equation (4) we get $0 = u - gt$, $\therefore t = u/g$.

2). *What is the greatest height reached?*

In equation (5) putting $t = u/g$ we get

$$s = u\frac{u}{g} - \tfrac{1}{2}g\left\{\frac{u}{g}\right\}^2 = \frac{u^2}{2g}.$$

We might get the same result more simply from equation (6). When the body ceases to rise, $v = 0$

$$\therefore \quad 0 = u^2 - 2gs. \quad \therefore \quad s = u^2/2g.$$

3). *When will the body return to the point of projection?*

The distance described *from the point of projection* in the required time is zero; hence from equation (5),

$$0 = ut - \tfrac{1}{2}gt^2, \quad \therefore \quad t = 0 \text{ or } 2u/g.$$

Comparing this result with 1), we see that the time taken for a body to fall from the greatest height reached, back again to the point of projection, is just the same as that taken by the body to reach its greatest height.

4). *What is the velocity of the body after returning to the point of projection?*

From equation (4). $v = u - g\left\{\frac{2u}{g}\right\} = -u,$

that is, the velocity is the same in magnitude as that at starting, but opposite in direction. Now, *since any point in the path might be considered a point of projection,* we infer from this result that the return or downward motion of the body is a *plane image* of the upward motion.

37. *The distances described in successive seconds (or other equal intervals of time) by a body, which starts from rest and is uniformly accelerated, are as the odd numbers.*

The distances described in 1, 2, 3, ... $(n-1)$, n seconds are $\frac{1}{2}a(1)^2$, $\frac{1}{2}a(2)^2$, $\frac{1}{2}a(3)^2$..............$\frac{1}{2}a\,(n-1)^2$, $\frac{1}{2}an^2$; $\therefore$ the distances described in the 1st, 2nd, 3rd.........nth seconds are $\frac{1}{2}a$, $\frac{3}{2}a$, $\frac{5}{2}a$,.............$\frac{1}{2}a(2n-1)$. Thus the distance described in the nth second, *where n is any number whatsoever,* is equal to $\frac{1}{2}a(2n-1)$, which varies as $(2n-1)$ the nth odd number.

Ex. A body is thrown vertically upwards with a velocity of 3922 tachs; find 1) the time taken to describe 58·83 metres, 2) the velocity at that height, 3) the greatest height reached, 4) the time of ascent, 5) the distance described in the half second following the fifth second from the instant of starting, 6) the distance described in 10 seconds.

1). Let t sec. = time required. From equation (5),

$$5883 = 3922\,t - \tfrac{1}{2}(980{\cdot}5)t^2, \therefore t = 2 \text{ or } 6.$$

2). From equation (4), velocity at the end of 2 seconds $= 3922 - 2\ (980{\cdot}5) = 1961$ tachs; velocity at the end of 6 seconds $= 3922 - 6\ (980{\cdot}5) = -1961$ tachs.

We thus see from 1) and 2) that the body in its *ascent* has risen 58·83 metres in 2 seconds; that after 6 seconds it is at the same height, but is then *descending;* that at both times the speed is the same.

3). From 2), art. 36, the greatest height reached

$$= \frac{(3922)^2}{2 \times 980{\cdot}5} = 7844 \text{ centimetres.}$$

4). From 1), art. 36, the time of ascent $= \dfrac{3922}{980{\cdot}5} = 4$ sec.

5). Distance described in $5\frac{1}{2}$ seconds

$$= 3922\ (\tfrac{11}{2}) - \tfrac{1}{2}(980{\cdot}5) \times (\tfrac{11}{2})^2$$

Distance described in 5 seconds $= 3922 \times 5 - \frac{1}{2}(980{\cdot}5) \times 5^2$

$\therefore$ the required distance

$$= 3922 \times \tfrac{1}{2} - \tfrac{1}{2}(980{\cdot}5) \times \tfrac{21}{4} = -612\tfrac{13}{16} \text{ cm.}$$

The $-$sign tells us that the body has *descended* down this distance in the 11th half second of its motion.

6). From equation (5) the distance required is

$$3922 \times 10 - \tfrac{1}{2}(980{\cdot}5) \times 10^2 = -9805 \text{ cm.}$$

The $-$sign tells us that the body is *below* the point of projection.

Examination IV.

1. Determine the equations of motion of a body uniformly accelerated in the direction of its motion.

2. Deduce the formula $v^2 = u^2 - 2as$.

3. A body starts from a given point with a velocity u, and has an acceleration a opposite in direction to u; determine 1) after what time will the velocity be zero? 2) After what time will the body return to the point of projection? 3) What is the velocity on returning to the point of projection? 4) What is the greatest distance travelled over?

4. Give the three equations of motion of a body let fall to the ground, neglecting the resistance of the air.

5. Prove that the distances described in successive equal intervals of time by a body, which starts from rest and is uniformly accelerated, are as the odd numbers.

6. Trace the motion of a body projected vertically upwards, and shew that the downward return motion is a plane image of the upward.

Exercise IV.

In the following examples the directions of velocity and acceleration are the same, and in the case of bodies moving vertically the resistance of the air is neglected. In all examples in the text-book take $g = 980{\cdot}5$ or $32\frac{1}{6}$ unless otherwise stated.

Log $980{\cdot}5 = 2{\cdot}9914476$, log $32\frac{1}{6} = 1{\cdot}5074061$.

1. A stone is observed to fall to the bottom of a precipice in 9 seconds; what is the depth? Given $g = 980$.

2. The height of the piers of Brooklyn Bridge is 277 feet; how long will a stone let fall from the top take to fall into the water?

3. A body is projected vertically upwards with a velocity of 320 vels. 1). How long will it rise ? 2). How far will it rise ? 3). When and where will its speed be 150 miles per hour ? 4). How long will it take to rise 1000 ft.? 5). What will its speed be at that height? 6). How far will it travel in the seventh second? Given $g = 32$.

4. A body starts with a speed of 1 metre per second, and has an acceleration of 10 tachs per second; what will its speed be after traversing $6\frac{1}{4}$ metres?

5. How long would a body which is projected with a downward velocity of 450 tachs take to fall through 15 kilometres, if there were no atmospheric resistance?

6. The speed of sound in air is constant, and at 10° C. is equal to 33833 tachs. The depth of the well in the fortress of Konigstein in Saxony is 195 metres. In what time should the splash of a stone dropped into the well be heard, if there were no atmospheric resistance ?

7. When a bucket of water is poured into this well, the splash is heard in 15 seconds; what is the *average* acceleration produced in the water by the resistance of the air?

8. A body whose acceleration is 10, traverses 6 metres in 10 seconds; what is the initial speed?

9. A body uniformly accelerated moves over 34·3 metres in the fourth second of its motion from rest: find the acceleration.

10. A person, starting with a velocity of 1 metre per second, and accelerating his velocity uniformly, traverses 960 metres in a minute; find his acceleration.

11. A body starts from a given point with a uniform velocity of 9 kilometres per hour: in an hour afterwards another body starts in pursuit of the first with a velocity of 2 metres per second, and an acceleration of 5 decatachs per hour; when and where will the second body overtake the first?

12. A body projected vertically upwards passes a point 10 metres above the point of projection with a velocity of 9805 tachs; how high will it still rise, and what will be its speed on returning to the point of projection?

13. A body uniformly accelerated describes 6·5 metres and 4·5 metres in the fourth and sixths seconds of its motion; find the initial speed and acceleration.

14. Two bodies uniformly accelerated in passing over the same distance have their speeds increased from a to b, and from c to d respectively; compare their accelerations.

15. Find the acceleration when in one-tenth of a second a speed is produced, which would carry a body over 10 metres every tenth of a second.

16. A particle is projected vertically upwards, and the time between its leaving a point 21 feet above the point of projection and returning to it again is observed to be 10 seconds; find the initial velocity. Given $g=32$.

17. Two bodies are let fall from the same place at an interval of two seconds; find their distance from one another at the end of five seconds from the instant at which the first was allowed to fall.

18. Two bodies let fall from heights of 40 metres and 169 decimetres reach the ground simultaneously; find the interval between their starting. Given $g=980$.

19. Two bodies start from rest and from the same point on the circumference of a circle; the one body moves along the circumference with uniform angular velocity about the centre. and the other, starting at the same time, moves along a diameter with uniform acceleration; they meet at the other extremity of the diameter; compare their speeds at that point.

20. A body, starting from rest with an acceleration of 20 tachs per second, moves over 10 metres; find the whole time of motion, and the distance passed over in the last second.

21. A body moves over 9 ft. whilst its velocity increases uniformly from 8 to 10 vels; how much farther will the body move before it acquires a velocity of 12 vels?

22. The path of a body uniformly accelerated is divided into a number of equal spaces. Shew that, if the times of describing these spaces be in *A.P.*, the mean speeds for each of the spaces are in *H.P.*

23. A body falling freely is observed to describe $24\frac{1}{2}$ metres in a certain second; how long previously to this has it been falling? Given $g=980$.

24. A body is dropped from a height of 80 metres; at the same instant another body is started from the ground upwards so as to meet the former half way; find the initial velocity of the latter body, and the speeds of the two bodies when they meet.

25. A body has a uniform acceleration a. If p be the mean speed, and q the change of speed in passing over any portion s of the path, shew that $pq=as$.

26. A body uniformly accelerated is observed to move over a and b feet respectively in two consecutive seconds; find the acceleration.

ANSWERS.

1. 396·9m. 2. 4·15. 3. 10; 1600; $3\frac{1}{8}$ or $16\frac{7}{8}$, $843\frac{3}{4}$; 3·9 or 16·1; 195·96; 112. 4. 150.
5. 54·9. 6. 6·9. 7. 793. 8. 10. 9. 980.
10. 50. 11. 4 h. 18 min. 59·8 sec.; 47849·6m.
12. 49025; 9904·5. 13. 1000, —100. 14. $b^2a^2 : d^2c^2$.
15. 10^5. 16. 164·15. 17. 7844. 18. 1.
19. $\pi : 4$. 20. 10; 190. 21. 11. 23. 2.
24. 2800; 2800,0. 26. $b-a$.

Chapter V.

Inertia. Mass.

38. *Inertia* is the inability of a body to alter its own condition of motion or rest. If a body be at rest, it remains so; if it be in motion, it goes on moving with the same velocity, *i.e.*, with constant speed in a straight line; and if it be rotating, it goes on rotating with the same angular velocity, about the same axis, which maintains a constant direction; *unless some other body interfere with it. To change the state of rest or motion of a body requires the presence of another body.* The term *force* is applied to the action of a body in altering the *status quo* of another body.

39. Inertia may be called a negative property, and yet it is one of the most obtrusive properties of matter. It is lucidly illustrated in railway and horse-riding accidents, in vaulting, jumping, and circus-riding, in shaking the dust from off a book, in the difficulty of driving over smooth ice, and in the action of a fly-wheel, which is used to regulate either an irregular driving-power, as in a foot-lathe, or an irregular resistance, as in a circular saw cutting wood. The tendency of bodies moving in circles, to fly off at every instant along the tangent, commonly but misleadingly called *centrifugal force*, is just inertia. Herein we have an explanation of the spheroidal form of the earth, and of the decrease of a body's weight, as we approach the equator. On letting a bullet fall from the top of a high tower or down a deep mine, it will, on account of its inertia, be found to fall somewhat to the east of the point vertically below that from which it fell, thus affording an ocular demonstration of the earth's rotation from *west* to *east*. The rotations of the earth and other mem-

bers of the solar system afford beautiful examples of inertia as regards rotation. The *constancy of direction* of the earth's axis, (except in so far as it is interfered with by the sun and moon) furnishes the most important step in the explanation of the changes of the seasons. By its inertia that interesting physical toy, the *gyroscope*, will prove that it is the earth and not the sphere of the heavens which daily rotates. The same principle, applied to the plane of oscillation of a pendulum, enabled Foucault to give one of the most convincing experimental proofs of the earth's rotation from west to east.

40. Sir Isaac Newton clearly enunciated the inertia of matter in his *First Dynamical Law:*

Every body remains in its state of rest or of uniform motion, except in so far as it may be compelled by impressed force to change that state.

In a scholium he referred to inertia as regards *rotation*. Here indeed there is a difficulty, for evidently the individual small particles of the rotating body move in circles, and must therefore be acted on by forces amongst themselves: else, on account of inertia, they would move in straight lines. However, when by *internal* forces the relative positions of the particles are fixed, the body will be as inert in its rotation as in its motion of translation.

41. Every body offers *resistance* to any change of its state of rest or motion. When the *same* force acts on different bodies it is found that the changes from the previous states of rest or motion are different, and this fact is expressed by saying that the bodies differ in *mass*. Mass, thus, is a property in which bodies may differ, just as they may differ in colour, volume, or weight. It might be defined as *the dynamical measure of a body's inertia*, or as *the capacity of a body to resist change of state of rest or motion.*

42. The difference in mass of different bodies (*e.g.* of balls of wood, ivory, lead, and iron, of different radii) may be lucidly illustrated by suspending the bodies by strings, and allowing the same spring, bent through the same angle, to act upon them in succession so as to give the bodies a horizontal motion. It will be found that the accelerations imparted will be very different.

The accelerations so produced would be the same if one of the bodies were at the surface of the moon, another at the sun's surface, and a third at the surface of Jupiter, where their weights would be respectively $\frac{1}{6}$th, 28 times, and $2\frac{1}{2}$ times as great as at the earth's surface.

43. How is mass measured? *When the same force is applied to different bodies, the masses of the bodies are defined as inversely proportional to the accelerations produced.*

Hence if the same force acts upon two bodies, and produces equal accelerations, the masses of the bodies are *defined* as equal to one another; but if the accelerations be in the ratio of $m:n$, the masses are *defined* to be in the ratio of $n:m$.

The C. G. S. unit of mass is called a *gram.* It is the mass of a cubic centimetre of water at 4° C (under the mean atmospheric pressure). The French unit of mass is the *kilogramme*, and is the mass of a litre of water at 4°C. The English and the F. P. S. unit of mass is a *pound* (avoirdupois). The pound was chosen perfectly arbitrarily. The present standard pound is a cylinder of platinum with a groove near one end. It is denoted as the P. S. or Parliamentary Standard, and is carefully preserved in London. The unit of mass is the third of the fundamental units in the C. G. S. and F. P. S. systems.

44. Let it be observed that by means of *one* force the masses of *all* bodies can be *theoretically* determined. When the same force acts upon bodies of the same mate-

rial, *e.g.* two pieces of iron at the same temperature, it is found that the accelerations are inversely as their volumes, (take as an illustration the opening of doors of the same kind of wood but of different sizes) ; but not so for bodies of different material, (take as an illustration the opening of a wooden and of an iron door). Hence it follows that *the masses of bodies of the same material (and at the same temperature and pressure) are directly proportional to their volumes, but not so for bodies of different material.*

45. These facts lead to the consideration of *density* and *specific mass.* The density of a substance is the mass per unit of volume. Hence in using C. G. S. units the density of water at 4° will be represented by 1. A cub. cm. of gold at 0° is found to be 19·3 grams, of rock-crystal 2·66 grams, of mercury 13·6 grams, of sea-water 1·027 grams, of dry air (under the mean atmospheric pressure) 0·0012932 gram. These facts are expressed by saying that the density of gold is 19·3, of rock-crystal 2·66, of mercury 13·6, of sea-water 1·027, of dry air 0·0012932.

46. The density of water, as of all other substances, varies with *temperature,* and (under the mean atmospheric pressure) is a *maximum* at 4° C. Hence it is that in defining unit of mass, the water is taken at this temperature. The density of water, as of all liquids, is very little changed by ordinary changes of pressure, so that it is hardly necessary to state, in defining unit of mass, that the water is supposed to be under the mean atmospheric pressure, the changes of atmospheric pressure making only *immeasurably small* changes of density.

47. The density of a body may be uniform, *i.e.* every part having the same density, or it may be variable. In the latter case we may speak of the density *at any point* of the body, or of the *mean* density of the whole body.

48. The *specific mass* of a substance is the ratio of the mass of any volume of the substance to the mass of an equal volume of water at 4°C. Whatever units be used, the specific masses of substances will evidently be represented by the same numbers, and with the C. G. S. units the density and specific mass of any substance will be represented by the same number. Hence the terms density and specific mass are frequently used indiscriminately, in the sense of specific mass, and almost always so when F. P. S. units are used.

The *specific volume* of a substance is the ratio of the volume of any mass of that substance to the volume of an equal mass of water at 4°C, and is evidently the reciprocal of the specific mass. The *rarity* of any substance or medium is the volume per unit of mass. Like density and specific mass the terms rarity and specific volume are generally used indiscriminately. Thus (Art. 45) the rarity or specific volume of dry air at 0° is 773·3.

49. Using C.G.S. units, the relation between the mass (m), the volume (V), and the density (d) of a body is given by the equation $m = Vd$. Using F.P.S. units, $m = 62{\cdot}4\ Vd$ expresses the same relation, since a cubic foot of water at 4° is 62·4 pounds nearly. In the above equations we see what an immense advantage the C. G. S. system has over the F. P. S. system of units.

50. The mass of a body is sometimes defined as *the measure of the quantity of matter in it*, or as *the dynamical measure of the quantity of matter in it.* Since we do not know the ultimate nature of matter this can hardly be scientifically correct. We only know the *properties* of matter, and can only measure its properties. Why then should quantity of matter be measured by one of these properties, *mass*, rather than by any other. We might reason thus: when the same quantity of heat is applied to bodies of the same substance, the changes of temperature

produced are inversely proportional to their volumes; but when applied to bodies of different substances, the changes of temperature are not inversely proportional to their volumes. We express this fact by saying that bodies differ in *thermal capacity*, and we define the thermal capacities as inversely proportional to the changes of temperature produced. Just then as with mass we might define the thermal capacity of a body as the *thermal measure* of the quantity of matter in it. We should then find the thermal measure and measurement by mass were quite different.

So long as we are dealing with bodies of *one* substance there are many ways in which we may measure quantity of matter quite intelligibly, *e.g.* by volume, by weighing in the same place, in the case of food by the length of time it will supply nourishment, in the case of fuel by the amount of water it will boil away, or by the amount of oxygen gas necessary for its complete combustion, and all these measurements would be found to agree with one another as well as with the measurement by mass. But when we come to deal with bodies of *different* substances, none of these measurements will be found to give results consistent with one another.

51. The reason doubtless why mass is stated to measure the quantity of matter in a body, is that this is the most familiar and most easily measured of the few properties of matter which remain measurably invariable through whatever changes the body may pass. Thus whilst by pressure, motion, heat, chemical action, or other agencies, we can easily alter the other measurable properties of a body, such as its volume, weight, elasticity, or thermal capacity, its mass, through whatever changes the body may pass, remains unchanged. This may be clearly illustrated by many experiments, *e.g.* by dissolving a piece of sugar in tea, by freezing a body of water, by mixing alcohol and

water, and, generally, in all chemical combinations. Whence the great law which forms the foundation of chemical science, the *Conservation of Mass :—Through whatever changes matter may pass, the total mass remains unchanged.* Hence the total mass of the universe is invariable.

Examination V.

1. Define inertia, and state the different forms thereof.

2. Give various illustrative examples of inertia.

3. What is centrifugal force ? Suggest a better name for it, and give illustrations thereof.

4. How may the earth's rotation from west to east be proved by ocular demonstration ?

5. Enunciate Newton's First Dynamical Law.

6. Define mass. How is it measured ?

7. Describe a simple experiment to shew difference of mass in different bodies.

8. Name and define the units of mass in the C. G. S. and F. P. S. systems of units.

9. What relation exists between the volumes and masses of bodies of the same material ? How is this proved ?

10. Define density, specific mass, specific volume, and rarity. Give the densities of a few common substances, and the rarity of air.

11. Why is water at 4° taken as the standard substance in measuring mass and density?

12. Give algebraical equations connecting the mass, volume, and density of a body. In the case of a body of variable density how do you express the relation?

13. Criticise the usual definition of mass as the measure of the quantity of matter in a body.

14. How did the above definition probably arise?

15. Enunciate the principle of the Conservation of Mass.

Exercise V.

1. A rectangular block of limestone is 2 metres long, 1·5 metre broad, and 1 metre thick. If 2·7 be its density, find its mass.

2. The sides of a canal shelve regularly from top to bottom. The width of a section at the top is 10 metres, at the bottom 5 metres, and the depth is 3 metres. If the canal be filled with water to a depth of 2·5 metres, find the mass of water per kilometre of length.

3. If the density of sea-salt is 2·2 and of sea-water 1·027, find the mass and volume of salt obtained in evaporating 100 litres of sea-water, if no contraction took place in solution.

4. The density of copper is 8·8, of zinc 7, and of brass formed from these 8·4; find the quantity of copper in 100 grams of brass.

5. The mass of a sphere of rock-crystal is 400·5, and its radius 3·3; find its density.

6. Find the mass of the earth, supposing it to be a sphere of radius 6371 kilometres, and of mean density 5·67.

7. Equal masses of copper and tin, whose densities are 8·8 and 7·3, are melted together; what would be the density of the alloy if no contraction or expansion took place?

8. When 63 litres of sulphuric acid, whose density is 1·84 is mixed with 24 litres of water, the volume of the mixture is 86 litres; find the mass and density of the mixture.

9. The mass of a nugget of gold-quartz is 350, and its density it 7·4; find the mass of gold in it. See art. 45.

10. The density of sea-water is 1·027; 100 litres of sea-water are frozen, and 20 kilograms of ice free from salt formed therefrom; what is the density of the residue?

11. What is the density of mercury, if 9 cubic inches have a mass of 4·42 lbs?

12. The density of milk is 1·03; how much water must have been added to 10 gallons of milk to reduce its density to 1·02.

13. From the summit of the Eiffel tower at Paris (lat. 48° 50′, $g=981$) a bullet is let fall 300 metres ; neglecting the resistance of the air, find how far to the east of the point, which is vertically under the point from which the bullet was dropped, it reaches the ground. Will the atmosphere increase or diminish the eastward deflection ? How ?

14. When a vessel is filled with equal volumes of two liquids, the density of the mixture is 9/8 of what it is when the vessel is filled with equal masses of the same liquids ; find the ratio of the densities of the two liquids.

15. Two liquids whose densities are as 1:2 are mixed together, (1) by masses in the ratio of the volumes of equal masses, (2) by volumes in the ratio of the masses of equal volumes ; find the ratio of the densities of the mixtures.

Answers.

1. 8100 kilogrs. 2. 17708·3 tonnes. 3. 4950; 2250.
4. 8·15. 5. 2·66. 6. 61418×10^{17} tonnes. 7. 7·98
8. 139920; 1·63. 9. 260. 10. 1·034. 11. 13·6.
12. 5 gals. 13. 11·23. 14. 1:2. 15. 18:25.

Chapter VI.

Momentum. Force.

52. The *momentum* of a particle is a property depending upon its velocity and mass, the direction of momentum being the direction of motion, and the magnitude of momentum being defined as proportional to the mass and speed conjointly. The term *quantity of motion* was used by Newton for momentum.

To vividly realize momentum let a person bathe close to a waterfall, say 200 ft. high, when he will feel the *drops of water*, which separate from the main mass, strike his body as if they were sharp stones. If he attempted to enter the main mass of falling water he would be roughly thrown on the ground.

53. In a scientific system of units, the unit of momentum is best defined as the momentum of a particle of unit mass moving with unit speed. Hence the C. G. S. unit of momentum is a *gramtach*, which is the momentum of 1 gram moving with a speed of 1 tach. Similarly the F.P.S. unit of momentum is a *poundvel*, the momentum of 1 pound moving with a speed of 1 vel. The unit of momentum evidently involves each of the three fundamental units of length, mass, and time, the first two *directly* and the third *inversely*.

The equation $M = mv$ expresses the numerical relation between the momentum, mass, and speed of a moving particle.

54. The time-rate of change of momentum at any instant will evidently be measured by the acceleration of the moving particle at that instant, and its mass conjointly, and is called the *acceleration of momentum*.

Force is that aspect of any external influence exerted on a body which is manifested by change of momentum. Whenever the momentum of a body changes, a force is said to act on the body.

If a definite change of momentum takes place in an immeasurably short time, (*e.g.*, when a cricket ball is struck by a bat), the action is called an *impulse*, the term *force* being usually applied when a finite time is required to produce a finite change of momentum, (*e.g.* when a body falls to the ground). All forces can be conceived to be made up of immeasurably small impulses, just as a curved line can be conceived to be made up of immeasurably short straight lines.

55. An impulse is measured in magnitude and direction by the whole change of momentum produced. A force is measured in magnitude by the acceleration of momentum, and its direction is the direction of the change of momentum. This is what Newton taught in his Second Dynamical Law:

Change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.

By *impressed* force Newton meant *external* to the body concerned. It is well in defining force to avoid the word *cause*. All that we are aware of is a change of momentum and the word force is conveniently used as a measure of the rate of this change. Under *energy*, one of the most important properties of matter, the student will learn that force may be defined as the space-rate of transference of energy, *i.e.* the rate of expenditure of energy per unit of length.

56. The *unit of force* is that force which produces unit of momentum per unit of time. Hence the C. G. S. unit of force is that force which produces 1 gramtach per second, or that force which acting upon a particle whose

mass is a gram gives it an acceleration of 1 tach per second. This is called a *dyne*. Similarly the F.P.S. unit of force is that force which produces 1 poundvel per second. or the force which acting upon a particle whose mass is a pound gives it an acceleration of 1 vel per second. and this is called a *poundal*.

When a force of f dynes or poundals acts upon a particle whose mass is m grams or pounds, and produces an acceleration of a tachs or vels per second, the equation which expresses the numerical relation between f, m and a is

$$f = ma$$

Force like velocity, acceleration, and momentum, is completely represented by a straight line.

57. *What does Newton's second dynamical law really teach?*

1). It defines the measurement of mass and force. Just as it is theoretically possible to measure *all* masses by means of *one* force, so is it possible to measure the magnitude of *all* forces theoretically by *one* mass. When *different* forces act upon the *same* body, the magnitudes of the forces are by definition directly proportional to the accelerations produced. It is indeed evident that *the mass of any body* is measured in grams by the reciprocal of the acceleration in tachs per second produced, when a dyne acts upon it ; and *any force* is measured in dynes by the acceleration in tachs per second produced, when it acts upon a body whose mass is a gram. 2). It enunciates the important experimental fact: *With whatever force different masses be measured, and with whatever mass different forces be measured, the measurements will always be alike.* 3). It asserts that the effect of a force depends in no way upon the motion of the body, and that when more than one force is acting on the body, each force produces its effect quite independently of the others.

58. It has been pointed out (Art 38), that whenever a force acts, there are always two bodies concerned. We generally speak of one of the two as receiving a change of momentum, and of the other as being concerned in the production of this change. Newton clearly pointed out in his Third Dynamical Law that the action was a mutual one; that change of momentum was received by both bodies, equal in magnitude but of opposite direction:

To every action there is always an equal and contrary reaction; or the mutual actions of any two bodies are always equal (in magnitude) *and opposite in direction.*

The word *force* is properly used when we consider the effect of the action between any two bodies in changing the momentum of one of them only. *Stress* is a term applied to the mutual action between any two bodies, when there is special reference to the *dual* character of that action, as enunciated by Newton. This third law tells us that all dynamical actions between bodies are of the nature of stresses. When a body falls to the ground under the action of its weight, the earth rises to meet it with an equal momentum. Since, however, the mass of the earth is so very much greater than that of any body on its surface, the motion of the earth is so small that it may be neglected. When two like magnetic poles, free to move, are brought near one another, it will be found that the repulsion is mutual. When the loadstone attracts a piece of iron, the iron attracts the loadstone with an exactly equal force. When the table is pressed by the hand, we feel that the hand is likewise pressed by the table. When a horse draws a canal boat by means of a stretched rope, the horse is drawn backwards with as great a force as the boat is drawn forwards. This may easily be proved by cutting the rope, when immediately the horse falls forwards. This is further seen, when we reflect that relatively to the boat the horse does not move at all. If two boats

are floating and one is drawn towards the other by means of a rope, the latter is also drawn towards the former with an equal momentum: *i.e.* the same rope is pulling both boats at the same time, and with equal force but in opposite directions. When two railroad trains or other bodies collide, the change of momentum in the one is just equal, and opposite in direction, to the change of momentum in the other whatever be the original direction or rate of motion of either train.

59. Since momentum has direction as well as magnitude, it at once follows from the above law, that the total momentum of two bodies is not altered by their mutual action. From this the important principle called the *Conservation of Momentum* is at once deduced:

The total momentum of any body, or system of bodies, cannot be altered by the mutual actions of its several parts.

As an illustration of this principle let us consider the *kick* of a gun. Here we have a system consisting of 3 bodies, the gun, the gas formed from the gunpowder. and the ball; it will be at once seen that the backward momentum of the gun is just the equivalent of the forward momentum of the ball.

The total momentum of the universe is a constant quantity is an immediate deduction from the same principle.

60. Different names are given to different aspects of force, such as *pressure, tension, attraction, weight, repulsion, friction.*

Pressure calls up the idea of *pushing.* This term is applied to a stress between particles close together. when the direction of each force is towards the particle acted upon, *e g.* the pressure of a fluid on the containing vessel.

Tension calls up the idea of *pulling.* This term is applied to a stress between particles close together when

the direction of each force is away from the particle acted upon. *e.g.* the tension of a stretched cord.

Attraction is a term applied to forces exerted between bodies, when there is no *sensible* material medium through which the force is exerted, and in consequence of which the bodies *approach* one another. The force between two unlike magnetic poles is a familiar case of attraction.

Weight, a well known form of attraction, is applied to the force exerted by the earth on any body at its surface. Forces are often conveniently measured by the weights of bodies of known mass. Thus, when a force of p grams is spoken of, a force equal to the weight of a body whose mass is p grams is meant. It would be better to speak of a force of p *grams-weight*.

Repulsion is a term generally applied to forces between bodies, when there is no *sensible* material medium through which the force is exerted, and in consequence of which the bodies *recede* from one another. The force between two like magnetic poles is a familiar example of repulsion.

Resistance is a term frequently applied to any force *opposing* the motion of a body, *i.e.* producing a *negative* acceleration. One of the most familiar and important of such resistances is the ubiquitous force of *friction*, a term applied to that force which is called into play, when one body moves or tends to move over the *surface* of another body. It is principally the force of friction which a locomotive works against in pulling a train along. The resistance which bodies experience in falling through the air is largely the force of friction between the bodies and the aerial particles they rub against.

61. In modern nomenclature the science of force is called *Dynamics*. It is divided into *Statics* and *Kinetics*.

Statics treats of *equilibrium* or the balancing of forces. It is chiefly concerned in determining the relations which must exist amongst a set of forces which keep a body at rest.

Kinetics investigates the forces acting on bodies having varying motions. The exact determination of the motions of the Solar System is the grandest problem in Kinetics, and is commonly known as Physical Astronomy.

Kinematics is the science of motion, when studied without any reference to mass. It forms an appropriate introduction to Kinetics. Chapters II., III., IV. belong to Kinematics.

Mechanics treats of the construction and uses of *machines*, and the relations of the forces applied to them. It forms the *practical* side of Dynamics.

Examination VI.

1. Define momentum ; name and define the unit of momentum in the C.G.S. and F.P.S. systems of units ; and write down the numerical relation between momentum, mass, and speed.

2. Define force and impulse, and give the measures and units of these in the two systems of units.

3. Enunciate Newton's Second Dynamical Law, and state fully all that it teaches. Express the law in algebraical form.

4. Enunciate Newton's Third Dynamical Law and give several illustrations thereof.

5. Enunciate and illustrate the Conservation of Momentum.

6. Define the terms pressure, tension, attraction, weight, repulsion, resistance, friction.

7. What is meant by a force of 10 pounds, or a force of 10 kilograms ? Give better expressions for these.

8. Define the terms Dynamics, Statics, Kinetics, Kinematics, Mechanics.

9. How can momentum be strikingly felt ?

Exercise VI.

1. A kilodyne acts upon a body at rest whose mass is 50 grams ; find the speed and distance passed over at the end of 10 seconds.

2. A body whose mass is 5, has an acceleration 2. At one instant the speed is 10 ; what is the momentum a minute afterwards ?

3. Find the acceleration produced by a megadyne acting on a solid sphere whose diameter is a decimetre and density 10.

4. A force of 50 kilodynes acts upon a body which acquires in 10 seconds a speed of a kilotach ; find the mass of the body.

5. The mean radius of the earth is 20,902,000 feet, its mean density 5·67, its mean distance from the sun 92·7 million miles, and the time of its revolution around the sun $365\frac{1}{4}$ days ; compare its momentum with that of a train of 10,000 tons, rushing along at 60 miles an hour.

6. The distance of Jupiter from the sun is 5·2 times that of the earth, its period $4332\frac{1}{2}$ days, its mass 310 times that of the earth ; compare the momenta of Jupiter and of the earth.

7. A body of 10 grams has a uniform acceleration of 10 m. per min. per min.; what force is acting upon it ?

8. A body acted on by a uniform force is found to be moving, at the end of the first minute from rest, with a speed which would carry it through 20 kilometres in the next hour ; compare the force with the weight of the body which would give it an acceleration $g=980·5$.

9. A kilogram is supported by a string 20 m. long. The string's mass is 2 grams per metre. Find the tension at the middle point, and at 5 decim. from the upper end.

10. Compare the momentum of a man, whose mass is 150 lbs. and latitude that of Kingston, Ont. (44° 13′), arising from the earth's rotation, with that of a steamship

of 10,000 tons, going at the rate of 15 miles an hour. A sidereal day $= 86{,}164{\cdot}1$ sec. See also ex. 5.

11. A body, acted on by a force of 100 kilodynes, has its speed increased from 6 to 8 kilometres per hour in passing over 84 metres; find the mass of the body.

12. A body of 1 kilogram is acted on by a uniform force in the direction of its motion, and is found to pass over 905 and 805 cm. in the 10th and 20th second of its motion from rest; find the force acting upon it and its initial speed.

13. Two bodies, acted on by equal forces, describe the same distance from rest, the one in half the time the other does; compare their final speeds and momenta.

14. Two bodies of equal mass uniformly accelerated from rest, describe the same distance, the one in half the time the other does; compare the forces acting on the bodies.

15. Two balls, one of silver and the other of ivory, whose diameters are as 1 to 2, are subjected to equal impulses; the speeds produced are as 22 to 15: compare the densities of silver and ivory.

16. If a ship be sailing with uniform velocity, what relation must exist between the driving force and the resistances of the air and water?

17. The density of lead is 11·4 and of cork 0·24. Two balls of these substances, whose diameters are as 1 to 10, are acted upon by equal forces during the same time; compare their momenta and speeds.

Answers.

1. 200; 10m. 2. 650. 3. 191. 4. 500.
5. 74937×10^{16}. 6. 1359:10. 7. 25/9.
8. 1:105·894. 9. 1020 grs.-wt.; 1039 grs.-wt.
10. 1:2685. 11. 77760. 12. 10 kilodynes; 1 kilotach.
13. 2:1; 1:2. 14. 4:1. 15. 60:11. 16. Equal.
17. Equal; 400:19.

Chapter VII.

Weight. Gravitation.

62. *Weight* is the attraction of the earth for every body on its surface, in virtue of which any body, unless it is supported, falls to the ground. It is also called the *force of gravity*. According to their weights bodies are called *heavy* or *light*. All bodies at the same place are found to fall in the same direction relatively to the surface of the earth, and bodies falling in contiguous places (*practically*, within a decametre of one another) fall in parallel straight lines. The direction in which a body falls at any place is called the *vertical* direction at that place, and is easily found by means of a *plumb-line*. Any direction at right angles to the vertical is called *horizontal*. The surface of any liquid (practically, within an *are* of area), at rest relatively to the earth, is a horizontal plane, except just where it meets the vessel containing it.

63. When different particles fall *at the same place* and *in vacuo*, the vertical acceleration is found to be the same for all. This is proved by the guinea and feather experiment. A simple but very instructive experiment which illustrates this fact, may be made by cutting out a piece of paper just sufficient to cover the mouth of a tin lid of a small box, and then allowing the lid and piece of paper to fall simultaneously from the same level, 1) when they are apart, 2) when the paper covers the mouth of the lid. Hence (Art. 56) we deduce the very important fact:

The masses of bodies are directly proportional to their weights at the same place.

$$w = mg,\quad g \text{ is constant},\quad \therefore\ m \propto w$$

64. The following extract from Lucretius, *De rerum natura*, shows that the fact, that all bodies would fall

equally fast *in vacuo*, was believed in, though not proved, 2000 years ago:

> In water or in air when weights descend,
> The heavier weights more swiftly downwards tend;
> The limpid waves, the gales that gently play,
> Yield to the weightier mass a readier way;
> But if the weights *in empty space* should fall,
> One common swiftness we should find in all.

65. Weight is measured like any other force in dynes or poundals. Thus the weight of a body whose mass is 1 gram is g or 980·5 dynes, and the weight of a pound of matter is g or $32\frac{1}{6}$ poundals. Forces are often conveniently measured in terms of the weights of known masses. Thus we read of a force of a kilogram-weight or a force of 10 lbs. wt., and these expressions are generally abbreviated into a force of a kilogram or a force of 10 lbs. The measure of a force in terms of weight is called its *gravitation* measure, that in dynes or poundals being called in contradistinction its *absolute* measure. Since g varies with latitude, it is evident that the gravitation measure of a force has not a definite value, unless the place be stated. The dyne or poundal on the other hand is *independent of place and time*, and is hence called an absolute unit.

66. The simple relation between the weights of bodies *at the same place* and their masses gives the best practical method of measuring the masses of bodies, as is done in a common balance. Observe that in a *common balance*, by comparing the weights of bodies with those of *standard* masses, we really measure *mass;* whereas in a *spring balance* we directly measure *weight*.

The law which explains to us the measurement of weight by means of a spring balance is known as Hooke's law: *The extension, compression, or distortion of a solid body, within the limits of elasticity, or the compression of*

a liquid, is directly proportional to the force which produces it.

67. The direct proportionality between the masses of bodies and their weights explains why mass and weight are constantly confounded with one another. The following illustrations in which these two properties of matter are contrasted, will assist the student to apprehend their difference:

1. *a*). The *mass* of a body is the same at whatever part of the earth's surface it be.

b). The *weight* changes with change of place, and is different at the Equator, at either Pole, or at the summit of the Rocky Mountains, from what it is in the class-room.

2. *a*.) The opening of a room door is essentially a question of *mass ;* and, however heavy the door may be, if the hinges are truly vertical and well oiled, a small child may open it, though slowly.

b). If the same door formed the lid of a box, and swung on horizontal hinges well oiled, the child could not open it, unless he had strength enough to exert muscular force equal to at least half the *weight* of the door.

In either case the child has to overcome the force of friction, which, though greater in the first than in the second case, is in either case small.

3. *a*). In moving a cart along a level road the horse has to exert a greater force at starting than afterwards, because he has to exert force to give the *mass* a given velocity, *i.e.* to produce *momentum*. After having started he has only to balance the force of friction.

b). When, however, he comes to a hill he has again to put forth his strength, for now he has, in addition to the force of friction, to overcome part of the *weight* of the cart.

4. *a*). The action of a fly-wheel, or of a small hammer, depends entirely upon its *mass*.

b). The action of a large steel hammer, such *e.g.* as the 125 ton hammer at Bethlehem, Pennsylvania, worked by steam, and used in shaping large and massive bodies, depends essentially upon its *weight*.

5. *a*). In athletic sports the "long jump" is essentially a question of *mass*.

b). In the "high jump" *weight* in addition has to be considered. Hence the actual distance of the high jump is never so great as that of the long jump.

6. *a*). In an undershot water-wheel the miller depends upon the momentum (and hence also the *mass*) of the running water to drive the wheel.

b). In an overshot water-wheel he depends upon the *weight* of the water which enters the buckets of the wheel.

68. How is g, the acceleration due to the force of gravity, at any place measured ? The most accurate method of finding this important physical quantity is by means of pendulum experiments. There is, however. one method of finding a fairly accurate value of g, which at this stage the student can understand. This is by means of the well-known physical instrument called Attwood's machine. The essential part of the apparatus is a grooved wheel which turns upon an axle, each end of which rests on two other wheels called the *friction wheels*, so that the friction on the axle of the first wheel is reduced to a minimum ; over this wheel passes a fine thread connecting two bodies of different weights. If m and m' be their masses, and m be the greater, the bodies will move on account of the greater weight of m with an acceleration equal to $(m - m')\,g \div (m+m')$, if we neglect friction and the masses of the wheels and thread. This acceleration can evidently be made as small as the experimenter pleases, by making the difference between m and m' small enough. By a clock and suitable adjuncts the acceleration of the moving bodies can be accurately measured, and therefore g approximately determined.

The following values of g at the sea-level have been determined by experiment and calculation :

	Latitude.	Value of g.
Equator	0° 0′	978·1
Sydney, N.S.W	33°51′	979·6
Tokyo	35°40′	979·8
Washington	38°54′	980·1
Rome	41°54′	980·3
Kingston, Ont.	44°13′	980·5
Paris	48°50′	980·9
London	51°30′	981·2
Berlin	52°30′	981·3
Edinburgh	55°57′	981·5
St. Petersburgh	59°55′	981·9
Pole	90° 0′	983·1

From the above values of g it is seen that the maximum variation over the earth's surface is about $\frac{1}{2}$ p.c. of the mean value.

69. The profound investigations of Sir Isaac Newton into the attractions between a few of the larger particles of matter in the universe, in order to explain the motions of the planets and their satellites, and especially the motion of the moon, the nearest neighbour in the universe to our own abode the earth, led this remarkable philosopher to the inevitable conclusion that it is the weight of the moon which keeps her revolving around the earth, and that weight is but a particular case of *gravitation*, which pervades the whole universe, and which acts according to the following law.

Law of Universal Gravitation: Every particle of matter in the universe attracts every other with a force, whose direction is that of the line joining the particles, and whose magnitude is directly as the product of their masses and inversely as the square of their distance.

Weight is thus only a particular case of a force which has been found to govern the motions of every body in the universe. Since Newton published his law to the world, its truth has been confirmed by every astronomer who has lived after him. By means of it, not only have the minutest perturbations in the motions of the heavenly bodies been rationally explained, but eclipses, transits, return of comets, and other heavenly phenomena have been predicted years before they took place, and actually did take place a few seconds within the times predicted. The nautical almanac, which guides our ships across trackless oceans, is but a book of predictions depending on the truth of this law. A great triumph in its application was the discovery of Neptune, the most distant planet of our solar system, which, though invisible to the naked eye, was discovered by mathematical calculations, which instructed the astronomer how to direct his telescope. Nor is this law confined to our own small solar system, but extends to systems millions of millions of millions of miles beyond our own, where not only does satellite revolve around planet, and planet around sun, but where one sun revolving around another has its motions governed by this same grand law.

70. We may express the law algebraically by the formula $f = G.\frac{m\,m'}{r^2}$, where m and m' are the masses of two particles, r their distance apart, and f the gravitation between them. G is a constant, and measures the gravitation between two particles, each of unit mass, and unit distance apart. To determine G, in C. G. S. measure, we may observe that, by comparing the attraction of the whole earth with that of a large leaden ball, and by other means, the mean density of the earth has been calculated to be 5·67. Hence, taking the earth's mean radius r to be 6,371 kilometres, and the mean attraction of the earth for a gram

of matter at the sea-level (allowing for centrifugal force) to be 982·3 dynes, we get

$$982{\cdot}3 = G.\frac{\frac{4}{3}\pi r^3 \times 5{\cdot}67}{r^2} = G\,\tfrac{4}{3}\pi \times 637{,}100{,}000 \times 5{\cdot}67$$

$$\therefore\ G = 6{\cdot}5 \times 10^{-8}$$

Hence two particles, one centimetre apart, and each of one gram mass, attract one another with a force of $6{\cdot}5 \times 10^{-8}$ dyne.

Examination VII.

1. Define weight. By what other name is it known? How is the direction of weight practically found?

2. How is it proved that g is the same for all bodies at the same place?

3. Define the terms vertical and horizontal, and give an illustration of each.

4. Prove that the weights of bodies at the same place are directly proportional to their masses.

5. Explain what is meant by an absolute unit of force, and express, in absolute units, forces of a pound-weight and of a kilogram-weight.

6. How would you prove to a person that the weight of a body is less, the nearer it is to the equator?

7. How are mass and weight practically measured?

8. Give illustrations of weight and mass, which contrast with one another, to shew the difference between these two properties of matter.

9. How is the value of g experimentally determined? Describe the essential parts of Attwood's machine.

10. Enunciate the law of universal gravitation, and shew how to determine approximately the gravitation between two grams of matter at the distance of a centimetre from one another.

11. Give the values of g at the Equator, Kingston Ont., Paris, and the North Pole, true to a decitach per second.

12. Enunciate Hooke's Law, and apply it to the spring balance.

13. If a merchant buys goods in London by means of a spring balance, and with the same balance sells in Kington Ont., will he gain or lose in the transaction? Why? By how much p.c.?

14. Show that a poundal is nearly half an ounce-weight; and that a dyne is nearly a milligram-weight.

Exercise VII.

1. A body whose mass is 10 grams is falling in vacuo; what is the force acting on it, and its momentum at the end of 10 seconds from rest?

2. A force of 50 grams weight acts upon a body which acquires in 10 seconds a speed of 39·22 tachs; find the mass of the body.

3. A force produces in a sphere of radius 10 cm. and density 10 an acceleration of 100 tachs per second; find what weight the force could balance.

4. If 250 lbs. be hung to the lower end of a rope 80 ft. long, find the tensions at the ends, the middle point, and 20 ft. from the upper end, the mass of the rope being 4 oz. per foot.

5. In Attwood's machine, if 10 kilograms be the mass of one body, and 15 kilograms that of the other; find the acceleration of momentum, and the speed at the end of two seconds.

6. A force of 12 pounds-weight acts upon a mass of 2 pounds; what is the speed after traversing a mile?

7. A mass of 10 pounds is acted on by a uniform force, and in 4 seconds passes over 200 feet; express in gravitation measure the force acting.

8. If the earth's mean radius be 20,900,000 feet and the mean attraction of the earth for a pound of matter at the sea level be 32·23 poundals, find the gravitation between two pounds of matter one foot apart.

9. In Attwood's machine one mass is known to be 10 lbs., and the distance described in 2 sec. is found to be 16 ft. 1 in.; find the other mass.

10. Find the diameters of two equal spheres of gold, such that the gravitation between them, when they just touch, is a dyne.

11. Find the unit of time, if a metre be the unit of length, a gram the unit of mass, and a gram-weight the unit of force, in a scientific system of units.

12. Find the unit of length, if a second, gram, and gram-weight be the units of time, mass, and force.

13. Find the unit of mass, if a second, centimetre, and gram-weight be the units of time, length and force.

14. A sphere of rock-crystal of density 2·66 has a diameter 6·5 cm.; find its volume, mass, and weight at Rome.

15. A body of 6 lbs. pulls by its weight another body of 4 lbs. along a smooth horizontal table; find the time taken to move through 965 ft. from rest, and the distance passed over in the last second.

16. Find the tensions of the three parts of a string, which supports at different heights bodies of 12, 6, and 4 lbs. respectively.

17. Oxygen combines chemically with hydrogen to form steam in the proportion of 8 parts by mass of oxygen to 1 of hydrogen. If the gases be weighed by means of a spring balance graduated at Edinburgh, how many milligrams-weight of oxygen at the Equator will combine with 100 milligrams-weight of hydrogen at Edinburgh to form steam?

18. Determine the mass of steam so formed, and its weight at Kingston Ont.. as indicated on the above spring balance.

19. Answer the above (17 and 18) when the gases are weighed in a common balance, and explain your answers.

20. If a kilogram be placed on a horizontal plane, which is made to descend vertically with an acceleration of 100 tachs per second; find in gravitation measure the pressure on the plane.

21. If 10 lbs. be placed on a horizontal plane, which is made to ascend vertically with an acceleration of 20 vels per sec.; find in lbs.-wt. the pressure on the plane.

22. If the speed of each of the bodies in Attwood's machine be 20 vels, when they are at the same height above the ground, and if at that instant the string be cut, find how far apart the bodies will be in 5 seconds.

23. If in ex. 20 and 21 the motions be vertical velocities of 100 tachs and 20 vels respectively, instead of accelerations, find the pressures on the planes.

24. One spring is stretched 2 cm. for every kilogram appended to it; another, 4 cm.; if 4 kilograms be appended to both, how far will they both be stretched?

Answers.

1. 9805; 98050. 2. 12500. 3. 4272 grams.
4. 250,270,260, and 265 lbs. wt. 5. 4902500; 392·2.
6. 1427. 7. 7·772 lbs.-wt. 8. $1{\cdot}04\times10^{-9}$ poundal.
9. 6 or $16\frac{2}{3}$ lbs. 10. 19·7 cm. 11. 0·32 sec.
12. 980·5 cm. 13. 980·5 grams.
14. 143·8; 382·5; 374955. 15. 10; 183·35.
16. 22,10, and 4 lbs.-wt. 17. 797·2.
18. 9 decigrs.: 899·1 milligrs. 20. 898·01 grs.-wt.
21. 16·22. 22. 200 ft. 24. $5\frac{1}{3}$ cm.

Chapter VIII.

Archimedes' Principle.

71. Since the weights of bodies at the same place are directly proportional to their masses, and since different bodies differ in their specific masses, therefore they will also have different *specific weights*, or, as they are also called, *specific gravities.*

The *specific weight* of a body is the ratio of its weight to the weight of an equal volume of water at 4° C (its maximum density point) *at the same place.*

Specific weight, being a ratio of quantities of the same kind, viz. weight, is like angle and specific mass merely a number, and is independent of all units.

The specific weight of water at 4° C will evidently be represented by unity, and it is evident that the numbers which represent the specific mass and specific weight of the same substance are the same.

In the case of a body whose specific weight is not uniform throughout, the above definition gives the *mean* specific weight of the body.

72. The most convenient methods of determining the specific weights of liquid and solid bodies depend upon the Principle of Archimedes:

Every body immersed in a fluid is subjected to a vertically upward pressure equal to the weight of the fluid displaced.

The truth of this principle is at once seen when we think that, if the body were replaced with a portion of fluid of the same kind without any other change, the weight of the fluid would be supported. Its truth is *sensibly* felt in bathing on a shingly beach, when it is found that, the deeper one enters the water, the less are

the soles of the feet hurt by the pressure of the stones on them. It can be proved directly by immersing in a liquid, a body, whose volume can be measured exactly (such as a cube, cylinder, or sphere). observing the apparent loss of weight of the body, and then weighing the amount of liquid displaced. In the case of a floating body, the weight of fluid displaced will be found to be equal to the entire weight of the floating body. Convenient experiments to show these facts are given in books on Experimental Physics.

According to Newton's Third Law (Art. 58), the fluid. on the other hand, is subjected to a vertically downward pressure equal to the weight of the fluid displaced. This can easily be shewn experimentally by balancing a vessel containing water in a common balance, and immersing in the water a body held by a cord. The equilibrium will be immediately destroyed, and the force necessary to restore equilibrium will be found to be equal to the weight of water displaced.

73. The occasion which led Archimedes to the discovery of this principle was the giving to him by King Hiero of Syracuse the problem:—to discover the amount of alloy which, the king suspected, had been fraudulently put into a crown, which he had ordered to be made of pure gold. It is said that Archimedes saw the solution of the problem one day on entering the bath, and probably it was by his observation of the *buoyancy* of the water. It may have been, however, by his noticing that the volume of the water which he displaced would just be equal, by the principle of impenetrability (Art 2), to the volume of the immersed part of the body. Indeed, one of the most important applications of the principle of impenetrability is to determine the volume of any irregularly shaped body, by immersing it in a liquid contained in a measuring glass, and noting the change of level which takes place.

74. Archimedes' principle is applied practically in many ways; *e.g.* in finding the volumes of irregularly shaped bodies like King Hiero's crown, in floating balloons in the air and iron ships in the sea, in lifting ships over bars formed at the mouths of rivers, in removing piles used in the construction of docks, as well as in determining specific weights, as explained in the following article.

75. There are three instruments used in finding specific weights accurately, viz. the *balance*, *hydrometer*, and *specific gravity bottle*. For less accurate values a *measuring glass* may be used, and for liquids, also *specific gravity beads*.

I. *Liquids, to an approximation of the first degree:*

a). By means of a balance.

Weigh a body which is not attacked either by water or the liquid, *e.g.* a piece of agate or a platinum ball, firstly in air, secondly in water, thirdly in the liquid whose specific weight is required:

Let w_1 = weight of the body in air,
w_2 = ... water,
w_3 = ... the liquid,

then s. w. of the liquid $= \dfrac{w_1 - w_3}{w_1 - w_2}$

b). By means of hydrometers.

These instruments, also called *areometers*, are essentially closed tubes, weighted at one end, for determining specific weights by observing how far they sink in water and other liquids, or by observing what weight will make them sink to a certain depth. The latter are called hydrometers of *constant* immersion, the former hydrometers of *variable* immersion.

1. By means of a hydrometer of constant immersion, *e.g.* Nicholson's.

Let w_1 = weight of the hydrometer in air.
w_2 = weight required to sink the hydrometer to the marked depth in water.
w_3 = ditto, ditto. ditto. in the liquid.

then s. w. of the liquid = $\frac{w_1 + w_3}{w_1 + w_2}$

2. By means of hydrometers of variable immersion.

These are called *salimeters* or *alcoholimeters*, according as they are used for liquids of greater or less specific weight than that of water. Both kinds have scales attached to them, which tell either the specific weight directly for any immersion, or the volume immersed, in which case the specific weight must be calculated. A thermometer is frequently attached to tell the temperature of the liquid.

c). By means of a specific gravity bottle.

Let w_1 = weight of the s. g. bottle empty.
w_2 = ... full of water,
w_3 = ... full of the liquid,

then s. w. of the liquid = $\frac{w_3 - w_1}{w_2 - w_1}$

II. *Solids, to an approximation of the first degree:*

a). By means of a balance.

Let w_1 = weight of the body in the air.
w_2 = ... water.

then s. w. of the body = $\frac{w_1}{w_1 - w_2}$

b). By means of a hydrometer of constant immersion.

Let w_1 = weight of the body in air.
w_2 = weight required to make the hydrometer itself sink to the marked depth in water.
w_3 = weight required to make the hydrometer, with the body attached to the lower part of it, sink to the marked depth in water.

then s. w. of the body = $\frac{w_1}{w_1 - w_2 + w_3}$

for evidently if w denote the weight of the hydrometer in air, then $w+w_2$ will be the weight of water displaced by the hydrometer, and $w+w_1+w_3$ the weight of water displaced by the hydrometer and body together: therefore the difference $w_1+w_3-w_2$ will be the weight of water displaced by the body alone : w_1 can easily be determined by the hydrometer, although it is simpler to measure it by means of a common balance. This method is useful in the case of bodies like cork which float in water.

c). By means of a specific gravity bottle.

This method is particularly convenient for finding the specific weights of powders.

Let w_1 = weight of the powder,

w_2 = weight of the specific gravity bottle, full of water,

w_3 = weight of specific gravity bottle, after the powder has been inserted, and the bottle thereafter filled up with water,

$$\text{then s.w. of the powder} = \frac{w_1}{w_1+w_2-w_3}$$

d). When a solid body is soluble in water, we find its specific weight relatively to a liquid in which it is insoluble, and multiplying this by the specific weight of the liquid, we get the specific weight of the body relatively to water. As an example let us take common salt, and adopt method *a*).

Let w_1 = weight of salt in air,

w_2 = weight in petroleum or turpentine, of a vessel to hold the salt.

w_3 = weight in petroleum or turpentine of the vessel containing the salt,

s = the s. w. of petroleum or turpentine,

$$\text{then s. w. of the salt} = \frac{w_1 s}{w_1+w_2-w_3}$$

76. The following propositions follow immediately from Archimedes' principle :

1). The mass of a floating body is equal to the mass of the displaced fluid.

2). When a body floats in a liquid, the volume immersed is to the whole volume as the specific weight of the body is to the specific weight of the liquid.

Examination VIII.

1. Define the specific weight of a body, and prove that the numbers which measure the density and specific weight of any body are the same.

2. Enunciate and prove Archimedes' principle.

3. Describe several illustrative experiments which prove the same principle, and state several practical applications thereof.

4. Explain why a boat built of *iron* can float in water. What is the s. w. of iron?

5. Give the history of the discovery by Archimedes of his principle.

6. What are the three chief practical methods of determining the specific weights of solid and liquid bodies?

7. Give formulae for all the methods in the case of both solid and liquid bodies.

8. What is a hydrometer? Give the names of the different kinds, and their respective uses.

9. When a solid body is soluble in water, how is its specific weight found?

10. How would you find the s. w. of sulphuric acid, sulphate of copper, sand, cork, paper, snow, ice, mercury?

11. Given a common balance with a hook to weigh bodies in water, a piece of cork, and a piece of lead sufficient to sink the cork in water; shew (giving a formula) how to determine the specific weight of the cork.

12. How can the volume of an irregularly shaped stone be accurately determined?

Exercise VIII.

1. A piece of limestone weighs 20·21 grams in air, and 12·82 in water; find its specific weight.

2. The s. w. of ice is 0·92, and of sea-water 1·027; find what fraction of an ice-berg is below the surface of the sea.

3. Compare 1) the mass of an ice-berg below the surface of the sea with that above, 2) the average depth of the ice-berg below the surface with the average height above.

4. A block of pine, the volume of which is 4 litres, 340 cub. cm., floats in water with a volume of 2 litres, 240 cub. cm. above the surface; find the s.w. of the pine.

5. A person whose mass is 150 lbs. enters the sea to bathe. If the s. w. of sea-water be 1·027, and of the human body 0·9, find the pressure on his feet when 5/6 of his body is immersed.

6. A ball of platinum whose mass is a kilogram, when in water weighs 955 grams; what will it weigh when in mercury (s. w. 13·6)?

7. A piece of iron (s.w. 7·5) floats in mercury; find what part of the iron is above the surface of the mercury.

8. When 1 lb. of cork is attached to 21 lbs. of silver, the whole is found to weigh 16 lbs. in water. If the s. w. of silver be 10·5, find that of cork.

9. A vessel, containing water, weighs 2034 grams; a kilogram of bronze (s. w. 8·4) is held in the water by a string; find what will now be the apparent weight of the vessel and water.

10. A piece of cork has s. w. $\frac{1}{4}$, and mass 534 grams; find the pressure necessary to keep the cork under sea-water, whose s. w. is 1·027.

11. A kilogram of lead, whose s. w. is 11·4, is suspended in water by a string; find the tension of the string.

12. Neglecting friction, with what acceleration will a silver ball (s. w. 10·5) sink, and an elm ball (s.w. 0·7) rise in sea-water (s. w. 1·027)?

13. Find what force would be necessary to immerse a kilogram of oak (s. w. 0·97) in mercury (s.w. 13·6.)

14. A body of 58 grams floats in water with two-thirds of its bulk submerged: find its volume.

15. A man whose mass is 68 kilograms can just float in fresh water; find the maximum weight he could bear up clear of the water, when floating in the sea (s. w. 1·027.)

16. How much lead (s. w. 11·4) will a kilogram of cork (s. w. $\frac{1}{4}$) keep from sinking in the sea (s. w. 1·027)?

17. A piece of hard wood of mass 7·6 grams is attached to the lower part of Nicholson's hydrometer, and it is then found that the force required to sink the hydrometer in salt water is just the same as before the wood was attached, viz., 12·6 grams-weight. If 1·03 be the s. w. of the salt water, find that of the wood.

18. A vessel quite full of mercury weighs 72·5 kilograms; a kilogram of iron is put into the vessel and held completely immersed in the mercury; what will now be the apparent weight of the vessel and contents?

19. If in ex. 18 the iron be fixed to a hook at the bottom of a vessel, what will be the weight of the vessel and contents? Explain the difference.

20. Answer 18, when the vessel is only partially filled with mercury, and no mercury is spilled in completely immersing the iron.

Answers.

1. 2·735. 2. 0·90. 3. 8·6; 2·05. 4. 15/31.
5. $7\cdot\dot{2}$ lbs.-wt. 6. 388 grs. 7. 61/136. 8. $\frac{1}{4}$.
9. 2153 grs. 10. 1660 grs.-wt. 11. 912 grs.-wt.
12. 884·6 and 458·0 tachs per sec.
13. 13 kilogrs.-wt. 14. 87. 15. 1836 grs.
16. 3415·7 grs. 17. 1·03. 18. 72·5 kilograms.
19. 71686·6 grs. 20. $74313\cdot\dot{3}$ grs.

Chapter IX.

Pascal's Principle. The Barometer.

77. Matter is divided into *solid* and *fluid*. A solid *e.g.* agate, is distinguished from a fluid in offering more or less resistance to change of form, a fluid offering little or none. Hence fluids under the action of weight must be kept in solid vessels. Fluids again are divided into *liquids* and *gases*. A liquid is a very incompressible fluid, and can be kept in an open vessel. Water, petroleum, and mercury at ordinary temperatures are liquids. A gas is a very compressible fluid, and must be kept in a closed vessel. inasmuch as it will fill any space into which it is admitted, even if that space be already occupied by gas. Air is a mixture of several gases, though principally of two, Nitrogen and Oxygen. When a gas is near its *point of condensation*, it is called a *vapour*. Aqueous vapor or steam is one of the components of air or the atmosphere, though the quantity is comparatively small and varies with time and place.

In elementary dynamics a solid is assumed to be *rigid*, *i.e.* that its parts maintain the same relative positions, whatever forces may be acting on it; a liquid is supposed to be incompressible; and a gas as obeying Boyle's law.

78. A fluid may be defined as a body which will change its shape, more or less quickly, under the action of any force however slight, until all force applied to it is normal to its surface at every point.

It is evident that a fluid cannot remain at rest under the action of any external pressure, unless pressure is applied normally at *every part* of the surface. Hence in measuring the action between a fluid and a surrounding body (which may itself be fluid of the same or of a different kind), it is necessary to get the pressure *per unit of*

area at each point of contact. Again when we think of the equilibrium of a small spherical particle of a fluid, whose centre is at a given point; and make the particle smaller and smaller, we see that *there is pressure in every direction at any point of a fluid.*

That the pressure at any point of a fluid is the same in all directions may be accepted as an experimental fact. It is deduced later on as a necessary consequence of the fundamental property of a fluid, viz. that *hydrostatic* pressure on any surface is normal to that surface.

79. *The pressure at any point of a fluid* is the pressure per unit of area on any plane surface containing the point, when the area of the plane is indefinitely diminished. Or,

The pressure at any point of a fluid is the pressure per unit of area which *would be* on any plane containing the point, if the pressure at every point of the plane were the same as at the point in question.

The systematic unit of fluid pressure (or of hydrostatic pressure, or of pressure–intensity in general) is unit of force per unit of area. Hence the C. G. S. unit of fluid pressure is 1 dyne per sq. cm. This is called a *barad.* The F. P. S. unit is 1 poundal per sq. ft.

80. The following important property of a fluid is generally known as Pascal's principle:

Pressure applied at the surface of a fluid contained in a closed vessel is transmitted without change to every particle of the fluid.

The student may compare this principle with the equality of tension at every particle of a stretched cord, due to forces applied at the ends of the cord. Various experimental illustrations can be found in books on experimental physics. It may also be deduced from the fundamental property of a fluid at rest (Art. 78). Hence, *if a fluid were at rest, and subject only to forces applied*

at its surface, the pressure would be the same at every point of the fluid.

81. Pascal's principle is admirably illustrated and most usefully applied in the *hydrostatic* or *Bramah press*, a machine by means of which great *mechanical advantage* is acquired. It consists essentially of two hollow cylinders connected by a tube. Water-tight pistons fit into these cylinders, whilst they and the connecting tube are filled with water or oil. Pressure applied to the smaller piston is transmitted through the liquid to the larger, with a mechanical advantage measured by the ratio of the areas of the bases of the pistons. The moving force is generally applied to the smaller piston through a lever which further increases the mechanical advantage gained. It was by means of this machine that the heavy tubes used in the construction of the Britannia bridge, which crosses the Menai Strait, were lifted into their places. The student will easily satisfy himself that *the principle of work* applies in this as in all other machines.

Let it be observed that Pascal's principle applies to any fluid whether *homogeneous* or not. Thus in the *piezometer* the principle is applied to measure the compressibilities of liquids, when in general the compressed fluids consist of two different liquids and a gas, The *Cartesian divers* is an amusing physical toy which illustrates in a unique manner the principles both of Pascal and Archimedes; also Boyle's law (Art. 94), and Art. 83.

82. Weight is a force which acts on every particle of a fluid and not merely at its surface. Hence, applying Pascal's principle, the following four propositions can be deduced:

In a fluid at rest under the action of no external force except that of weight, surfaces of equal pressure are horizontal planes.

In large fluid bodies, like one of the oceans or the atmosphere, surfaces of equal pressure are at each place perpendicular to the direction of weight at that place, and are nearly spherical surfaces having small curvature.

83. *In a liquid of uniform temperature under the action of no external force except that of weight, the pressure increases uniformly with the depth.*

Take a vertical right cylinder of the liquid, the area of each horizontal base being s, its length h, and the density of the liquid d. It is evident that the pressure on the lower base must be greater than that on the upper by the weight of the cylinder, *i.e.* in C.G.S. units by $s\,h\,d\,g$ dynes. Hence the increase of fluid pressure per unit of depth is $g\,d$ barads, or d grams-wt. per sq. cm. In F.P.S. units the increase of fluid pressure per unit of depth will be $62{\cdot}4\,gd$ poundals per sq. ft., or $62{\cdot}4\ d$ pounds–wt. per sq. ft.

This proposition was experimentally illustrated by Pascal by bursting a large barrel by means of a long fine column of water. It shows us that in the supply of water from a reservoir to the houses of a town, the pipes far below the level of the reservoir need to be much stronger than those near the level of the reservoir. In Barker's mill and many applications of turbine-wheels the principle is taken advantage of to produce motion in machinery.

84. *When two or more fluids of different specific weights, which do not mix with one another, are put into the same vessel, they arrange themselves in the order of their specific weights, and their surfaces of separation, when the fluids are at rest, are horizontal planes.*

This is an immediate corollary of the two preceding articles. It may be illustrated by putting mercury, water, benzine, and air into the same bottle. In this, as well as in the following article, let it be observed that just where the surfaces of separation meet the containing vessel, they

are not horizontal on account of the action of the *external molecular force* of *adhesion.*

If a globule of oil be put into a mixture of water and alcohol, having the same s.w. as the oil, it will assume the form of a sphere under the action of internal molecular force.

85. *The free surface of a liquid at rest, under the action only of weight, and the pressure of the atmosphere, is a horizontal plane.*

This is just a particular case of last article, and is lucidly illustrated by putting a liquid into a series of communicating vessels of different shapes. The adage "water will always find its level" is a popular way of expressing the truth of the proposition. The principle is most usefully applied in supplying the houses of cities with water, in water and spirit levels, and in the construction of fountains. It explains the action of Artesian wells.

If two liquids of different specific weights be contained in different vessels, whose bottoms communicate by means of a tube completely filled with the denser of the liquids (art. 84), the free surfaces cannot be in one plane, but will be at heights above the common surface of separation which are inversely proportional to the specific weights of the liquids (art. 83.)

86. If A denote the pressure of the atmosphere, and d the density of a liquid, the pressure on any horizontal plane of area S at a vertical distance z below the free surface of the liquid is $(A+gdz)\,S$, by arts. 80 and 83.

The pressure does not depend in any way upon the form of the vessel, but only upon z and S.

This is beautifully illustrated by the famous experiment of Pascal's vases.

87. The student must not confound the pressure on the bottom of any vessel containing a liquid, with the pressure

resulting from the presence of the liquid on the body supporting the vessel. The latter is just the weight of the liquid, whilst the former may be greater or less according to the shape of the vessel. This fact is commonly called the *hydrostatic paradox*. It is easily understood when it is noticed that the pressure of the vessel on the body supporting it is the weight of the vessel together with the *resultant* of the pressures of the liquid on the whole interior surface of the vessel, and not merely the resultant of the pressures on the bottom.

88. The pressure of the atmosphere at any place is measured by the length of a vertical column of mercury which it can support, as shown by Torricelli in 1643 in his famous experiment:—Fill with mercury a glass tube which is closed at one end and is about 90 centimetres long and 1 centimetre in diameter; prevent air from entering the tube by placing a finger over the open end; put the open end into a vessel of mercury and remove the finger; it will then be found that the mercury will sink in the tube till the level inside is about 76 centimetres above the level of the mercury outside. This column of mercury Torricelli conclusively proved was supported by the pressure of the atmosphere. It was the first *barometer*, and at the present day is the most perfect barometer which can be made.

89. The pressure of the atmosphere changes both with time and place. The mean value over the earth's surface *at the sea level* is the same as would be produced by the weight of a vertical column of mercury 76 cm. long, at 0°C., in the latitude of Paris. The pressure per sq. cm. is therefore equal to the weight at Paris of 76 cub. cm. of mercury at 0°, *i.e.* (since 13·6 is the density of mercury at 0°) $13·6 \times 76$, or 1033·6 grams–wt. at Paris, *i.e.* (since 980·9 is the value of g at Paris) $980·9 \times 13·6 \times 76$ barads, or 1·014 megabarad.

90. *The pressure of the atmosphere is produced by its weight.* This was conclusively proved in 1648 by Pascal. Amongst other experiments he sent a Torricellian barometer to the top of the Puy de Dome, and found that there, the length of the mercurial column was considerably shorter than it was at the bottom of the mountain, just as he had predicted. It was fully 9 centimetres shorter. As in the case of liquids (art. 83), it is evident that at any altitude the length of Torricelli's column can be affected only by the weights of the aerial particles at higher levels. Hence as we ascend the mercury falls.

The very great pressure of the atmosphere (at the sea-level, nearly 14·7 lbs.–wt. per sq. inch, over 1 ton–wt. per sq. ft., and over 10 tonnes–wt. per sq. metre) may be strikingly shewn by means of the Magdeburg hemispheres. It is taken advantage of in a useful manner in the suction-pump, siphons, pipettes, and other appliances. The limiting height (about 33 feet or 10 metres) to which water can be raised by means of a *suction-pump* was accidently discovered in 1640 by some Florentine workmen, and it was this discovery which first led Galileo to suspect that the pressure of the atmosphere was due to its weight. The very name *suction*-pump recalls the old explanation of the rise of the water, by means of the long abandoned axiom "Nature abhors a vacuum."

91. *Hydrodynamics* is the dynamics of fluids, and is divided into *hydrostatics* and *hydrokinetics*. *Pneumatics* is a term frequently used to denote the dynamics of gases, and *hydraulics* the science which treats of machines for the conveyance of water or other liquids. *Archimedes' screw* is one of the oldest of hydraulic machines. It is used for raising water or any other liquid from one level to another, in a most ingenious manner, by taking advantage only of the weight and fluidity of the liquid particles. It consists of a tube wound round a cylinder into a helix

or screw. If the axis of the cylinder be inclined to the vertical at a greater angle than the angle of the screw, and the lower end of the tube be immersed in water, the water can be raised by rotating the screw-tube about the axis of the cylinder. Such machines were used in ancient Egypt for draining the land after inundations of the Nile.

Examination IX.

1. Define the terms solid, fluid, liquid, gas, vapor, rigid, and point of condensation.

2. Give a full definition of fluid, sufficient for the study of hydrostatics.

3. Explain the statement that the pressure at any point of a fluid is the same in all directions.

4. Define the pressure at any point of a fluid, and give the C.G.S. and F.P.S. units of fluid pressure.

5. Enunciate and explain Pascal's principle. Give an important practical application thereof.

6. Enunciate and prove four important propositions regarding the action of weight on a fluid at rest.

7. Explain the construction of a Cartesian diver. What important principles does it illustrate?

8. Explain the bursting of Pascal's barrel. Give important applications of the same principle.

9. Calculate the mechanical advantage of an hydrostatic press. What is meant by saying that the principle of work applies to it?

10. What is the form a fluid would take under the action of internal molecular forces alone? How would you shew this experimentally?

11. How would you illustrate experimentally the adage, "water will always find its level." State several important practical applications.

12. What does the experiment of Pascal's vases prove? What is the hydrostatic paradox? Explain it.

13. Describe Torricelli's experiment whereby he first measured the pressure of the atmosphere.

14. State in gravitation and in absolute measure the mean sea-level atmospheric pressure over the earth's surface.

15. How did Pascal prove that the pressure of the atmosphere was due to weight?

16. Describe the experiment of the Magdeburg hemispheres.

17. Explain the actions of the suction-pump, siphon, and pipette.

18. If the barometer be inclined at an angle i to the vertical, what is the length of the mercurial column?

19. A siphon is filled and held with its legs pointing downwards and the ends closed; what will take place, when a) one end is opened, b) both ends are opened, 1) when the ends are in the same horizontal plane, 2) when they are not in the same horizontal plane.

20. How does a change of atmospheric pressure affect the pressure between a liquid and the containing vessel?

21. Does a change of atmospheric pressure affect the action of a siphon, or of a suction-pump?

22. Examine the effects of making a small aperture in different parts of the barometer-tube.

23. A small piece of glass gets into a barometer and floats; is the reading vitiated thereby?

Exercise IX.

1. Find in lbs.-wt. per sq. in. at what rate the pressure increases per 10 ft. of vertical distance from the reservoir in the water-pipes of a town.

2. The deepest sounding taken on the "Challenger" was 8184 metres between the Carolines and Ladrones in the N. Pacific ocean. Find at that depth the pressures in atmospheres, and in tonnes-wt. per sq. cm., taking 1·027 as the mean s. w. of the water.

3. The legs of a siphon are equal in length and inclined at an angle i; how should the siphon be placed so as to remove the most liquid?

4. Find the greatest height over which sulphuric acid (s.w. 1·84) can be carried by a siphon when the barometer is at 30 inches.

5. If a mercurial barometer 1 sq. in. in section stand at 30 inches. what will be the height of a sulphuric acid barometer of section 1/1·84 sq. inch?

6. If the coefficient of cubical dilatation of a barometer tube were the same as that of mercury. would the height of the mercurial column be affected by change of temperature? Explain your answer.

7. The scale of a barometer is etched on the glass tube and is true at 17°. The readings of the barometer and thermometer are 75·67 and 10°. Find the reduced barometic pressure, *i.e.* the length of the Torricellian column when the temperature of the mercury is 0°; given k_3 (coefficient of cubical dilatation) for mercury to be $1{\cdot}8\times10^{-4}$, and k_1 (coefficient of linear dilatation) for glass to be 8×10^{-6}.

8. If the temperature at Edinburgh be 15°, and the scale of the barometer ($k_1=1{\cdot}9\times10^{-5}$) be true at 0°, find the reading of the barometer when the atmospheric pressure is a megabarad.

9. A siphon-barometer is held suspended in a vessel of water by a string attached to its upper end. If h denote the depth of the upper mercurial surface, and A gram-wt. per sq. cm. the atmospheric pressure, and s the internal section of the tube, find the difference of level of the two mercurial surfaces. At what rate does the tension of the string change as the barometer is lowered?

10. A string can bear a tension of 10 kilograms wt.; determine how much cork (s.w. $\frac{1}{4}$) it can keep below the surface of mercury (s.w. 13·6).

11. Two bodies of 1 and 2 kilograms are attached to a string passing over a smooth pulley; the bodies rest in equilibrium when they are completely immersed in water; if the specific weight of the first body be 2, find that of the second.

12. A specific gravity bottle when entirely filled with distilled water has a mass of 530 grains; 26 grains of sand are put into the vessel, and the whole then weighs 546 grains: find the specific weight of the sand.

13. Two liquids which cannot mix are poured into a circular tube so as to occupy a quadrant each; the diameter joining the free surfaces is inclined at $\frac{1}{3}\pi$ to the vertical: find the ratio of the densities of the liquids.

Answers.

1. $1\frac{1}{3}$. 2. 813; 8405.
3. Legs equally inclined to the vertical.
4. 221·7 inches. 5. 18 ft. 5·7 inches. 6. Yes.
7. 75·54. 8. 75·10.
9. $(A+h)\div 12{\cdot}6$: $s/25{\cdot}2$ grams-wt. per cm.
10. 187·3. 11. 4/3. 12. 2·6. 13. 3·73.

Chapter X.

Specific Weights of Gases.

92. We are aware from the effects of wind in driving windmills, ships, &c., that air has mass. That it has weight like solid and liquid bodies can easily be proved by the following experiments :

Exp. 1. Weigh a globe with a tightly fitting stop-cock, firstly full of air, secondly after the air has been extracted from it by means of an air-pump.

Exp. 2. Boil water in a flask until all the air is driven out, cork it up tightly, weigh when cool, admit air and weigh again.

Exp. 3. Instead of extracting air from the globe in exp. 1, compress air into it, when it will be found to become heavier.

Exp. 4. Weigh the globe in exp. 1 when filled, firstly with air, secondly with hydrogen, thirdly with carbonic acid.

The second and third experiments are due to Galileo; from the third the specific weight of air may be approximately measured by collecting the compressed air in a pneumatic trough. The fourth experiment proves that gases like liquid and solid bodies, differ in specific weight.

93. The fact that gases have weight, and even flame, which is essentially incandescent gas, was known to the Epicureans, if we take Lucretius as their mouth piece. In his great poem, "De rerum natura," written about 56 B.C., he says:

> See with what force yon river's crystal stream
> Resists the weight of many a massy beam;
> To sink the wood, the more we vainly toil,
> The higher it rebounds with swift recoil:

Yet that the beam would of itself ascend,
No man will rashly venture to contend:
Thus too the flame has weight, though highly rare
Nor mounts but when compelled by heavier air.

94. Before considering how the specific weights of gases have been determined, it will be necessary to know how the density of a gas depends upon its pressure. The physical law which tells us this is called Boyle's law, and may be enunciated in either of the following ways:

The density of a gas is directly proportional to its pressure, if the temperature be far above the point of condensation, and remain constant. $d \propto p$.

The volume of a gaseous body is inversely proportional to the pressure, if the temperature be far above the point of condensation, and remain constant. $pV = c$.

The truth of Boyle's law depends of course entirely upon experimental evidence. Dry air is an example of a gas far above its point of condensation, and for dry air at all ordinary pressures and temperatures the law may be said to be exact. When the temperature approaches the point of condensation, the product pV gradually decreases. In the case of solid and liquid bodies it is not necessary to consider the pressure to which they are subjected. for the changes of density, arising from the changes of pressure to which bodies are in general exposed, would be immeasurably small. The case of gases is very different.

95. It was the great French physicist, Regnault, who first overcame the experimental difficulties necessary to an exact determination of the specific weights of gases. The secret of his success depended upon counterbalancing the globe containing the gas he was weighing, with another globe of equal volume and weight as nearly as possible, so that it was unnecessary for him to make any corrections for barometric, thermometric, or hygrometric changes in

the state of the atmosphere during the time of experimentation. Having several times exhausted one of these globes and filled it with a dried gas until he was satisfied that the globe was thoroughly dry, he put the globe into a mixture of ice and water (0° C), and filled it once again with the dried gas at the pressure of the atmosphere, say P cm. of mercury. He then partially exhausted the globe, to pressure p say, keeping it at the same temperature 0° C, and noted the change of weight. This change of weight (w) will, by Boyle's law, be the weight of the gas at 0° which would fill the globe at pressure $P-p$; therefore the weight of gas at 0° required to fill the globe at the mean atmospheric pressure will be $76\, w \div (P-p)$.

In this way Regnault found the weights of equal volumes of dry air and other gases. It remained for him to determine the specific weight of dry air with respect to the standard substance, water at 4° C. If w' denote the difference of weight between the globe when filled with water at 0°, and when filled with dry air at 0° and pressure P, then $w' + \{w\, P \div (P-p)\}$ will denote the weight of the water which the globe would hold at 0°. If therefore s denote the s. w. of water at 0°, the s.w. of dry air at 0° and under the mean atmospheric pressure will be

$$\frac{76\, w\, s}{P-p} \div \left(w' + \frac{wP}{P-p}\right)$$

If it be necessary to take into consideration the buoyancy of the air in determining w and w', the methods indicated in the next chapter will explain how this can be done. The following table gives the results of some of Regnault's experiments:

	Mass of 1 litre at 0° and 76 cm.	Specific volume.
Air (dry)	1·293187	773·3
Oxygen	1·429802	699·4
Nitrogen	1·256167	796·1
Carbonic Acid	1·977414	505·7
Hydrogen	0·089578	11.163

By *76 cm.*, or *a pressure of 76 cm.* the mean atmospheric pressure is always meant, (Art 89).

96. On account of the small densities of gases it is generally more convenient to measure their specific weights with respect to dry air or hydrogen as a standard, than with respect to water at 4° C.

The specific weight of a gas with respect to dry air (or hydrogen), is defined as the ratio of the weight of any volume of the gas at 0° C and under the mean atmospheric pressure, to the weight at the same place of an equal volume of dry air (or hydrogen) at the same temperature and pressure.

The following table gives the specific weights of the above gases with respect to dry air and hydrogen:

Hydrogen	0·0693	1·000
Nitrogen	0·9714	14·023
Air (dry)	1·0000	14·436
Oxygen	1·1056	15·962
Carbonic Acid	1·5291	22·075

97. The numbers in the last column and similar results have enabled chemists to establish a most important law, called Avogadro's law, which may be enunciated in the three following ways:

The specific weights of gases, at the same temperature and pressure, are directly proportional to their molecular weights, if the temperature be far above the points of condensation.

The molecular volumes of gases, at the same temperature and pressure, are all equal to one another, if the temperature be far above the points of condensation.

The number of molecules in a gaseous body, at a given pressure and temperature, the temperature being far above the point of condensation, is directly proportional to the volume, and is independent of the nature of the gas.

In the following table for gases the specific weights have been calculated from the molecular weights, with the exception of those of hydrogen, nitrogen, air, and oxygen, which are Regnault's experimental measurements.

Table of Densities and Specific Weights.

I. *Solids at* 0° *C.*

Platinum, stamped	22·10	Sapphire	4·01
" rolled	22·07	Diamond	3·52
" cast	20·86	Glass	2·5 to 3·3
Gold, stamped	19·36	Kingston Limestone	2·70
" cast	19·26	Rock-crystal (Quartz)	2·66
Lead, cast	11·35	Ice	0·92
Silver, cast	10·47	Ivory	1·92
Copper, hammered	8·88	Anthracite	1·80
" cast	8·79	Ebony, American	1·33
Bronze, Brass, } Average	8·40	Mahogany, Spanish	1·06
		Box, French	1·03
Steel	7·82	Oak, English	0·97
Iron, wrought	7·79	Maple, Canadian	0·75
" cast	7·21	Elm	0·70
Tin, cast	7·29	Willow	0·58
Zinc, cast	7·00	Poplar	0·38
Aluminium	2·67	Cork	0·24
Magnesium	1·74	Pith, of sun-flower	0·028

II. Liquids at 0° *C.*

Mercury	13·596	Water at 4°	1·000000
Sulphuric Acid	1·84	" at 0°	0·999873
Human Blood	1·05	Olive Oil	0·92
Milk, of Cow	1·03	Alcohol	0.80
Sea-water	1·027	Ether	0·72

III. Gases at 0°*C and* 76 *cm. pressure.*

Hydrogen	0·0000896	Nitrogen	0·0012562
Ammonia	0·0007619	Air (dry)	0·0012932
Aqueous Vapour	0·0008044	Oxygen	0·0014298
Steam at 100°	0·0005887	Carbonic Acid	0·0019658
Carbonic Oxide	0·0012510	Chlorine	0·0031684

98. By a good air-pump hydrogen can be rarified to a density 10^4 times less than what it has under the atmospheric pressure. We thus see, by comparing the density of platinum with that of rarified hydrogen, the great range of density there is, even in substances which can be easily obtained. It will be seen from the above table that the density of a solid depends to a certain extent on the way in which it has been prepared. Even in the case of natural bodies like sapphires and diamonds, different specimens from different places are found to vary slightly in density. In mixtures the density is not always the mean of the densities of the component parts. Thus bronze has a greater density than the mean of the densities of the component metals; so with a mixture of alcohol and water. In the case of woods different parts of the same tree vary in density, as well as specimens from different trees of the same species. Liquids can be obtained more easily in a state of purity, but in such liquids as blood, milk, and sea-water, differences of density are found in different specimens.

Examination X.

1. How do we become aware that air possesses the property of mass? Describe three experiments to prove that air has weight.

2. Enunciate Boyle's law in two ways, and show that the one follows from the other.

3. What was the secret of Regnault's success in weighing gases? Describe fully his method of finding the specific weight of dry air.

4. Define the specific weight of a gas with respect to dry air, and also with respect to hydrogen, and give the s.w. of dry air, 1) with respect to water at 4°, 2) with respect to hydrogen.

5. Enunciate Avogadro's law in three ways.

6. What is the range of density as found by experiment.

Exercise X.

1. Determine the mass and weight at the equator of 10 litres of oxygen at 0°, and at a pressure of 74 cm. of mercury at 0° at the equator.

2. Determine the pressure in barads under which chlorine has a density 3 with respect to air at 0° and 76 cm.

3. A balloon is filled with hydrogen, the pressure and temperature being 76 cm. and 0°. If the capacity of the balloon be a megalitre, and the non-gaseous material have a mass of 500 kilograms, and a mean density of 1, find with what acceleration the balloon will begin to ascend.

4. A flask of 2 litres capacity was found to weigh 1·6 grams more, when filled with carbonic acid, than when filled with air at 0°; find the pressure of the atmosphere.

5. Find what the volume and density of a litre of air at 0° and 76 cm. will become at the bottom of the deepest known part of the ocean (s.w. 1·027), viz. 8184 metres, if the air obeyed Boyle's law to that depth.

6. A uniform tube, 1 m. long, closed at one end, is 3/4 full of mercury, and is then inverted as in the Torricellian experiment into a vessel of mercury; if the barometic column be 75 cm., find what will be the height of the mercury in the tube.

7. A uniform tube, 1 m. long, open at both ends, is immersed in a vessel of mercury to a depth of 90 cm. If the top be now closed, and the tube raised, until the length out of the mercury be 90 cm., find the height of the mercury within the tube, 75 cm. being the length of the barometric column.

8. What will be the length of tube out of the mercury in last example, when the air within the tube is 30 cm. long?

9. A barometer was carefully calibrated, and it was found that on account of a small bubble of air getting into

the tube, the reading was 75 cm., when the true pressure was 76 cm., and that the space occupied by the air was equivalent to 10 cm. of tube; what would be the true pressure, at the same temperature, when the reading was 76 cm., and what would be the reading when the true pressure was 75·4 cm.?

10. If 75 cm. be the height of the mercurial barometer, find how far a conical wine-glass must be immersed mouth downwards in water, so that the water may rise half-way up in it, 7 cm. being the length of the axis of the cone, and 13·5 the density of the mercury relatively to the water.

11. Prove that the atmosphere must be at least 5 miles high. Is this true from whatever level you measure?

12. A cylindrical diving-bell, 3m. high, is lowered to the bed of a river 12m. deep. If 75 cm. be the barometrical pressure, find the height of water in the bell, and compare the mass of air in the bell with what must be forced in to keep the water out.

13. A diving-bell is suspended at a fixed depth; a man who has been seated in the bell falls into the water and floats; find the effect on 1) the level of the water in the bell, 2) the amount of water in the bell, 3) the tension of the chain holding the bell.

14. The receiver of an air-pump is 4 times as capacious as the barrel. Shew that after 3 strokes the pressure of the enclosed air is reduced to nearly $\frac{1}{2}$, and that it takes another 3 to reduce the pressure to $\frac{1}{4}$. Given log. $2=0·3010300$.

15. The gauge of a condensing pump consists of a glsss tube containing air, whose volume is determined by the position of a drop of mercury C in the tube. If A be the position of the mercury when the air in the condenser is uncompressed, and B the end of the tube, prove that if air is forced uniformly into the condenser, the ratio $AC : CB$ increases uniformly.

16. A condenser and suction-pump have the same barrel and receiver, the capacity of the barrel being $\frac{1}{10}$ of that of the receiver. Ten strokes are given to the condenser; how many strokes must be given to the suction-pump, so that the pressure of the air in the receiver may be that of the atmosphere. Given log. $11 = 0{\cdot}0413927$.

17. Two barometers of the same length and the same section are immersed in the same reservoir, and each contains a small quantity of air; their readings at one time are d, e, and at another time h, k; if l denote the length of each tube above the surface of mercury in the reservoir, shew the quantities of enclosed air are as

$$\frac{d-e}{l-k}-\frac{h-k}{l-e} : \frac{d-e}{l-h}-\frac{h-k}{l-d}$$

18. A gas contained in a cubical vessel is compressed into the sphere which can be inscribed in the cube, shew that the *total* pressures on the two confining surfaces are equal. If the gas be allowed to expand until it fills the sphere circumscribing the cube, shew that the total pressure on the confining surface is lessened in the ratio $\sqrt{3} : 1$.

19. A cylinder with a closely fitting piston is full of gas; shew that, if the piston be pressed into the cylinder at a constant speed, the total pressure on the base increases harmonically, whilst the total pressure on the cylindrical surface remains constant.

20. A water-tap is connected with a mercurial siphon-manometer, and on opening the tap the difference of level of the surfaces of mercury was found to be 4 ft. 3 in.; find the head of water, and the available water-pressure.

ANSWERS.

1. 12·556; 12281. 2. $1{\cdot}2414 \times 10^6$. 3. 1171·6.
4. 90·4 cm. 5. 1·23, 1·05. 6. 42·4. 7. 54·1.
8. 80. 9. $77\frac{1}{9}$; 74·45. 10. 7091.
12. 152; 51/60. 13. 1) raised, 2) lessened, 3) lessened.
16. About $7\frac{1}{4}$. 20. 57·8 ft.; 39·7 lbs.-wt. per sq. in.

Chapter XI.

Exact Specific Weights.

99. Since the principle of Archimedes evidently applies to gases as well as to liquids. all bodies in the atmosphere are subjected to a vertically upward pressure equal to the weight of the air displaced by them. This may be illustrated experimentally by the *baroscope* and *balloons.* In determining the specific weights of solid and liquid bodies to an approximation of the first degree, we neglected the buoyancy of the surrounding atmosphere. Let us now determine these specific weights to an approximation of the second degree. This is done by taking into consideration the buoyancy of the air, but, without noting what may be its barometric, thermometric, and hygrometric states, taking as its s.w. the mean s.w. of the atmosphere at the place where the weighings are made. A good average value is 0·0012. The following problem will illustrate the process.

100. *Given w_1, w_2, w_3, the number of grams which balance a solid body in air, water, and another liquid respectively; to determine the specific weights of the solid body and liquid to an approximation of the second degree.*

Let r denote 0·0012 gram-weight. The approximate volume of the solid body will be $(w_1 - w_2)$ cub. cm., and therefore the weight of air displaced by the solid body will be $(w_1 - w_2)\, r$ gram-weight approximately.

Let s denote the s. w. of the standard masses against which the body is weighed. This should be determined by the maker of the standard masses. Then w_1/s is the volume of the standard masses which balance the body in air, and therefore $(w_1/s)\, r$ the approximate weight of air

in grams displaced by them. Therefore the approximate weight of the solid body in vacuo

$$= w_1 - (w_1/s)\,r + (w_1 - w_2)\,r \text{ grams-weight} = W_1$$

The approximate weight of the solid body in water

$$= w_2 - (w_2/s)\,r \text{ grams-weight} = W_2$$

The approximate weight of the solid body in the liquid

$$= w_3 - (w_3/s)\,r \text{ grams-weight} = W_3$$

Then the specific weight of the solid body $= \dfrac{W_1}{W_1 - W_2}$

and.. .. ..liquid body $= \dfrac{W_1 - W_3}{W_1 - W_2}$

each to an approximation of the second degree.

101. To get the specific weight of a body to an approximation of the third degree, we require to calculate the density of the air at the pressure, temperature, and hygrometric state in which it is at the time the weighings are performed, as well as to allow for the temperature of the water in which the body is weighed. We have already learned from Boyle's law (art. 94) how the density of a gas depends upon its pressure. The law of change of density of a gas, arising from change of temperature, was first discovered by Charles. It may be enunciated thus:

The dilatation of a gas, at temperatures far above its point of condensation, and at a constant pressure, is directly proportional to the increase of temperature; and the coefficients of dilatation are the same for all gases.

If the dilatation be reckoned from 0° C, the coefficient of dilatation is very approximately 0·003665 or 1/273. Hence if V_t be the volume at temperature $t°$, and V_0 the volume at 0°, we may express the law algebraically thus:

$$V_t = V_0\,(1 + 0{\cdot}003665\,t), \text{ or } V_t = V_0\,(1 + t/273)$$

If now temperature be reckoned from the zero of the air thermometer, *i.e.* from —273°C, Charles' law may be expressed thus:

The volume of a gas at temperatures far above its point of condensation, is, at a constant pressure, directly proportional to its temperature reckoned from the zero of the air thermometer. Or thus:

The density of a gas at temperatures far above its point of condensation, is, at a constant pressure, inversely proportional to its temperature reckoned from the zero of the air thermometer. Or thus:

The pressure of a gas at temperatures far above its point of condensation, is, at a constant volume, directly proportional to its temperature reckoned from the zero of the air thermometer.

102. Boyle's and Charles' laws can be conjointly expressed algebraically by either of the following equations:

$$\frac{p\,v}{t}=c, \text{ or } \frac{p}{d\,t}=k$$

in which p, v, d, and t denote the pressure, volume, density, and temperature *reckoned from the zero of the air thermometer.* The quantities c and k are constant, so long as we keep to the same gaseous body. If v', d' denote its volume and density at 0°C and 76 cm. pressure, then

$$c=\frac{76\,v'}{273}, \text{ and } k=\frac{76}{273\,d'}$$

103. To allow for the hygrometric state of the air, we require first to know the law of Dalton relating to the pressure of a mixture of gases. It may be enunciated thus:

When two or more gases, which do not act chemically on one another, are enclosed in a vessel, the resultant pressure is the sum of the pressures of the gases when placed singly in the vessel.

The physical principle underlying Boyle's and Dalton's laws has been beautifully expressed by Rankine thus: *When one, or more gases, which do not act chemically on one another, is confined in a vessel, each portion of gas, however small, exerts its pressure quite independently of the presence of the rest of the gas in the vessel.*

104. Boyle's, Charles', and Dalton's laws can be conjointly expressed algebraically by either of the following equations:

$$\frac{PV}{T} = \Sigma\left\{\frac{pv}{t}\right\} = \frac{76\,V'}{273}, \text{ or } \frac{P}{DT} = \Sigma\left\{\frac{p}{dt}\right\} = \frac{76}{273D'}$$

in which p, v, d, t denote the pressure, volume, density and temperature of any one of a number of gases to be mixed together; P, V, D, T, denote the pressure, volume, density, and temperature, of the mixture; and V', D' denote the volume and density of the mixture at the standard temperature and pressure, 0° C and 76 cm. T and t must be reckoned from the zero of the air thermometer.

105. The amount of aqueous vapor in the atmosphere varies with time and place, whilst the other constituents are found in almost unvarying proportions. Dalton's law tells us that the pressure of the moist air is just the sum of the pressures of the dry air and of the aqueous vapor mixed with it. From the classical experiments of Regnault the pressure of the aqueous vapour in the atmosphere can be determined, as soon as the *dew-point* is known. The dew-point is the temperature at which the atmosphere at any place would be *saturated* with the aqueous vapour which it contains. It is found experimentally by means of a *hygrometer*.

The following is a part of Regnault's table of the maximum pressures (or pressures of saturation, or pressures of condensation) of aqueous vapour at different temperatures. It gives the pressure of the aqueous vapour in the atmosphere, in millimetres of mercury at 0° at the latitude of Paris, for dew-points from 0°C to 29°C.

°C	mm.	°C	mm.	°C	mm.	°C	mm.	°C	mm.	°C	mm.
0	4·6	5	6·5	10	9·2	15	12·7	20	17·4	25	23·6
1	4·9	6	7·0	11	9·8	16	13·5	21	18·5	26	25·0
2	5·3	7	7·5	12	10·5	17	14·4	22	19·7	27	26·5
3	5·7	8	8·0	13	11·2	18	15·4	23	20·9	28	28·1
4	6·1	9	8·6	14	11·9	19	16·3	24	22·2	29	29·8

Observe, that whilst the pressure of a gas *at a temperature far above its point of condensation* depends upon its *temperature and volume*, the pressure of the same gas *at its point of condensation* (or, *in contact with its own liquid*) depends upon its *temperature alone*.

106. The following example will illustrate how the density of the atmospheric air can be calculated when its barometric, thermometric and hygrometric states are known.

Ex. The reading on the barometer is 76·4, the temperature 20°, the dew-point 8°, and the latitude 44° 13′ (Kingston, Ont.); to determine the density of the air, given the coefficient of dilatation of the barometer scale, which is true at 0°, to be 0·000008, and the mean coefficient of dilatation of mercury between 0° and 20° to be 0·00018.

The barometric pressure in centimetres of mercury at 0° in the latitude of Paris will be

$$=\frac{76{\cdot}4\ (1+20\times0{\cdot}000008)}{1+20\times0{\cdot}00018}\times\frac{980{\cdot}5}{980{\cdot}9}=76{\cdot}11.$$

This pressure is due, according to Dalton's law, partly to dry air, and partly to aqueous vapor in the air. According to Regnault's tables the pressure of the aqueous vapor for the dew-point 8° is 0·80 cm. Therefore the pressure of the dry air in the atmosphere is 75·31 cm. Hence applying Boyle's and Charles' laws, the density of the dry air in the atmosphere

$$=\frac{1{\cdot}2932}{1000}\times\frac{75{\cdot}31}{76}\times\frac{273}{293}=0{\cdot}001194,$$

the density of the aqueous vapour in the atmosphere

$$=\frac{8{\cdot}044}{10000}\times\frac{0{\cdot}80}{76}\times\frac{273}{293}=0{\cdot}000008$$

∴ the density of the atmospheric air $=1{\cdot}202\times10^{-3}$.

107. *To determine the specific weight of a solid and of a liquid body to an approximation of the third degree.*

Let w_1, w_2, w_3, denote the number of grams which balance the solid body in air, distilled water and the liquid respectively; the reading of the barometer 76·4, the temperature 20°, the dew-point 8°, and the latitude 44° 13′; the coefficient of dilatation of the barometer scale 0·000008. the mean coefficient of dilatation of mercury between 0°, and 20°. according to Regnault, 0·00018, and the density of distilled water at 20°, according to Despretz. 0·998213.

Find, as in art. 100, W_1 the approximate weight of the solid body in vacuo, S the s. w. of the solid body of an approximation of the second degree, and, as in last article, R the density of the air. Denote by s the s. w. of the standard masses against which the body is weighed, and by S' the s. w. of distilled water at 20°.

The weight of the solid body in vacuo (in grams) is *very* nearly $= w_1 + (W_1/S - w_1/s)\ R = W'$

The weight of the body in distilled water at 20°

$$= w_2 - (w_2/s)\ R = W''$$

The weight of the body in the liquid at 20°

$$= w_3 - (w_3/s)\ R = W'''$$

Then the s. w. of the solid body at 20° $= \dfrac{W'}{W' - W''} S'$

and the s. w. of the liquid at 20° $= \dfrac{W' - W'''}{W' - W''} S'$

each to an approximation of the third degree.

If the coefficients of dilatation of the solid body and liquid be known, the specific weights at any other temperature may be determined. By taking the specific weight of the solid body just determined in place of S, and W' in place of W_1, and repeating the method above, we could find the specific weights to an approximation of the fourth degree and so on to higher degrees. This would however be useless, as the errors of experimentation would certainly be greater than any errors, from the exact values, of the specific weights to an approximation of the third degree.

Examination XI.

1. How can it be proved experimentally that Archimedes' principle applies to gases? Explain the rise of smoke in the air.

2. Given the weights of a solid body in the air, water, and another liquid, to determine the specific weights of the solid body and liquid to an approximation of the second degree.

3. Enunciate in four ways the law of Charles, and deduce each one from the others.

4. Enunciate Dalton's law relating to the pressure of a mixture of gases. Give Rankine's statement of the physical principle underlying Boyle's and Dalton's laws.

5. What are the various corrections to be made in determining the specific weight of a body to an approximation of the third degree? What are the physical instruments used for this purpose?

6. Define the dew-point. What does it tell us?

7. Write down an algebraical equation which expresses conjointly the gaseous laws of Boyle, Charles, and Dalton.

8. A merchant buys against lead standard masses, and sells against aluminium standard masses. Does he gain or lose thereby? By how much p.c?

9. Which is more favourable for purchasers of goods, a high or a low barometer?

Exercise XI.

1. The reading of the barometer in a room is 77·34, the thermometer 15°, the dew-point 10°, the latitude 44° 13′; the coefficient of expansion of the barometer scale is 0·000008, and the mean coefficient of expansion of mercury between 0° and 15°, according to Regnault, 0·00018; the room is 12·5m. long, 5·45 m. broad, and 3·7 m. high; find the volume, mass, and weight of the air in the room.

2. A lump of gold weighs 437·008 grams in air, 414·357 in distilled water, and 420·699 in ether; the reading of the barometer is 77·3 cm., of the thermometer 9°, the dew-point 4°; the latitude, that of London; the s. w. of the standard masses against which the body is weighed is 8·4, the coefficient of linear expansion of the barometer scale 0·000018, the mean coefficient of expansion of mercury between 0° and 9°, according to Regnault, 0·00018, and the density of distilled water at 9°, according to Despretz, 0·999812; to determine the specific weights of gold and ether to approximations of the first, second, and third degrees.

3. A cubic decimetre of aluminium just balances a lump of lead when both are in water; which will weigh the heavier in air? Why? Find their difference in weight, 1) in vacuo, 2) in air (s. w. 0·0012), 3) in air as indicated by a common balance, the s. w. of the standard masses being 8·4.

4. Find the force necessary to hold down a balloon, of which the capacity is 150000 litres, when filled with hydrogen, the pressure and temperature being 77 cm. and 15° C., and the weight in the air (s. w. 0·0012) of the solid material of the balloon being 14·5 kilograms.

5. Two hollow spheres (radii 1:2) contain equal masses of air at 10° and 20° respectively; compare 1) the pressures of the gases, 2) the total pressures on the spheres.

6. Find at what temperature the density of dry air is 0·01 at pressure 76 cm., and at what pressure the density is 1 at temperature 17° C?

7. Compare the densities of the air at the top and bottom of a mine shaft, the temperatures being respectively 11° and 18°, and the pressures 74 cm. and 77 cm.

8. The temperatures at the bottom and top of the mountain Fuji in Japan were respectively 20° and 4°, and

the reduced barometric pressures 76·2 cm. and 48 cm. Shew that the density of the air at the top of the mountain was just about $\frac{2}{3}$ of that at the bottom.

9. Ten litres of oxygen at 74 cm. and 18° are mixed with 5 litres of hydrogen at 75 cm. and 15°; find the pressure of the mixture when the volume is 10 litres, and temperature 0°.

10. Compare the volumes of hydrogen at 0° and 77 cm., and of oxygen at 20° and 74 cm., which will be in proper proportion to form steam (H_2O) which consists of 8 parts by mass of oxygen to 1 part of hydrogen.

11 A vessel filled with hydrogen contains some water, and the pressure at 0° is found to be 76 cm.; find what the pressure will be at 20°, when the volume of the gas is reduced one-half.

12. The Torricellian vacuum is 40 cm. long, and 1 sq. cm. in section, the temperature 0°, and the atmospheric pressure 76 cm. Find what would be the height of the mercurial column if there was admitted into the vacuum 1) a centigram of dry air, 2) a milligram of hydrogen, 3) a decigram of water, 4) a gram of ether, all at 0°. Given the *pressure of condensation* of ether at 0° to be 18·4 cm.

ANSWERS.

1. 252,062,500; 312,084·6; 306 megadynes.
2. 19·293, 0·72001; 19·271, 0·72035; 19·267, 0·72023.
3. 838·65 grs.–wt.; 837·64 grs.–wt.; 837·76 grs.–wt.
4. 152,592·4 grs.–wt. 5. 2264:293; 566:293.
6. –238°C or 35°A; 821 atmospheres.
7. 197:200 nearly. 9. 105 cm. 10. 1·786:1.
11 163·89 cm. 12. 64·57; 60·67; 75·53; 57·59.

Chapter XII.

Work. Energy.

108. *Work* is the production of motion against resistance. *Energy* is the power to do work. Work is physically manifested either in accelerating the motions of bodies, or in changing the configuration of a material system. Thus when a man lifts a body up, he does work against the body's weight, and by the work done produces a change of configuration of the earth. Again, when he throws a cricket ball, he does work in giving the ball motion.

We have firstly defined work, then energy. The order might have been reversed, thus: *Energy* is the power to overcome resistance through space; *work*, the expenditure of energy, or the transference of energy from one body to another.

By the *configuration* of a material system at any instant is meant the condition of the system as regards the relative position of its several parts. The term *form* refers particularly to the bounding surface of any body or system of bodies. Configuration refers to all parts of the body or system, whether internal or on the bounding surface.

109. A body in motion has energy in virtue of its mass and speed (art. 112), and this is called *kinetic energy*. Such is the energy of a cannon ball, which enables it to tear down a rampart against the resisting molecular forces. Similarly, in virtue of its mass and speed, a running stream can drive a water-wheel and thus grind our corn. These are examples of *molar kinetic energy*.

110. A body may also possess energy in virtue of its mass, and of its position with respect to other bodies, and this is called *potential energy*. It is found by experience

that there are forces which act between every pair of particles in the universe. The force of gravitation, the molecular forces (cohesion, elasticity, crystalline force, &c.), the atomic force or chemical affinity, are different aspects of such force. When work is done against such force upon a body which forms a part of a material system, so as to alter the configuration of that system, the body in virtue of its new position has energy which it did not previously possess. Thus a *head* of water has energy in virtue of its position with respect to the earth. The *wound up* spring of a clock can keep it going for a week or longer. *Compressed* air, such as is used for the conveyance of letters in large cities, is a store of energy in virtue of the configuration of the aerial particles. These are examples of *molar potential energy*.

A material system, such *e.g.* as the solar system, possesses molar energy; firstly, on account of the motions of its component parts; this is its kinetic energy; secondly, on account of its configuration; this is its potential energy. An oscillating pendulum, or a vibrating spiral spring, is a beautiful and simple example of a body whose molar energy is constantly passing from the one form into the other. At the extremities of the line of vibration, the energy is wholly potential; at the middle point, it is wholly kinetic; and at intermediate positions it is partly kinetic and partly potential. In an undershot water-wheel the miller depends upon the kinetic energy of the water to grind his corn; in an overshot water-wheel, upon the potential energy of the water to drive the wheel.

111. Work is measured by the force overcome and the distance through which it is overcome conjointly. Thus in measuring the work done in raising bricks to the top of a house, the builder multiplies the weight of the bricks by the vertical height through which they are raised. To raise double the number of bricks through double the height will evidently require four times as much work.

The *unit of work* is that in which unit of force is overcome through unit of distance. In the C.G.S. system the unit of work is the work of overcoming a dyne through a centimetre, and is called an *erg*. The equation $w = fs$ evidently gives the relation between the work done in ergs, the force overcome in dynes, and the distance in centimetres through which the force is overcome.

112, *To determine the kinetic energy of a body whose mass is m and speed v.*

Let the body move against a uniform resistance f. This will give the body an acceleration f/m. opposite in direction to the body's motion. If s be the whole distance through which the body can act against this resistance, so that after passing through the distance s the speed is zero, by equation (6) art. 35,

$$0 = v^2 - 2(f/m)s, \therefore fs = \tfrac{1}{2}mv^2$$

but fs is evidently the total work done by the body against the resistance f, and is all that it can do, since. after doing this work, the speed is zero. Hence $\frac{1}{2}mv^2$ measures the body's kinetic energy, or the amount of work the body can do, in virtue of having mass m and speed v. If m be measured in grams, and v in tachs, $\frac{1}{2}mv^2$ measures the body's kinetic energy in ergs. Since the kinetic energy of a body varies as the *square* of its speed, it is evident that it is independent of direction; in this respect it is well to note, energy differs from momentum and force.

113. As an illustration of the preceding article let us consider the case of a body of weight w and mass m, thrown vertically upwards in vacuo with speed u. In virtue of its kinetic energy it raises itself against its own weight. If h be the greatest height reached. the work done is wh. Now $w = mg$ (art. 63), and $h = u^2 \div 2g$ (art. 36), therefore $wh = \frac{1}{2}mu^2$, which, as might be expected, is the same result as we got in last article.

Is the energy of the body in its elevated position destroyed? No, it is merely in a latent form; for, without

imparting any more energy to the body, we can get out of it, *in virtue of its new position*, the same amount of work as it was capable of doing at starting. This will be at once understood when we remember that by letting the body fall to its point of starting, it acquires the same speed which it had at starting (art. 36), and has therefore again the original kinetic energy imparted to it. In its elevated position the energy of the body is *potential*. Such is the energy of a head of water used to drive machinery, or of the elevated heavy bodies whose energy is used to drive piles into the ground.

We see from the above that the potential energy of an elevated body, relatively to the earth, is measured by wh, where w is the body's weight, and h its height above the ground. If w be measured in dynes and h in centimetres, then wh measures the potential energy in ergs. Also from art 35, equations (3) and (6), it is easily seen, that in any intermediate position of the body between the ground and height h, the energy of the body is partly kinetic and partly potential, and that the total energy is constant and equal to wh or $\frac{1}{2}mu^2$.

114. Just as it is convenient in many practical questions to have a gravitation as well as an absolute unit of force, so in the practical measurement of work it is often convenient to use a gravitation unit. Such a unit is the *kilogrammetre*, or work done in raising a body of 1 kilogram vertically upwards against its weight through the height of 1 metre. Evidently 1 kilogrammetre $=10^5 g$ ergs.

115. The F.P.S. unit of work is that required to overcome a poundal through the distance of a foot, and may be called a *foot-poundal*. English engineers use as a gravitation unit of work a *foot-pound*, *i.e.* the work done in raising 1 pound vertically upwards through the distance of 1 foot. The foot-pound is evidently equal to g or nearly $32\frac{1}{6}$ foot-poundals.

If m be the mass of a body in pounds and v its speed in vels, then $\frac{1}{2}mv^2$ measures its kinetic energy in foot-poundals (art. 112), and therefore $\frac{1}{2}mv^2 \div g$, its energy in foot-pounds. Similarly if a body, whose mass is m pounds and weight w poundals, be at a height of h feet above the earth's surface, it has potential energy measured by wh foot-poundals, *i.e.* mgh foot-poundals, or mh foot-pounds.

116. The *unit rate of working*, or the *unit of activity* in the C.G.S. system is 1 erg per second. If H denote the rate of working in ergs per second, f the resistance in dynes, and v the speed in tachs of the body moved against the resistance, then $H = fv$. This formula suggests the name *dyntach* for the unit of activity. Watt's *horse-power* is a convenient gravitation unit adopted by English engineers, and is equal to 550 foot-pounds per second. The French *force-de-cheval* is a similar gravitation unit equal to 75 kilogrammetres per second, or nearly $7{\cdot}36 \times 10^9$ dyntachs. These were supposed to be rates at which a good horse works, but are now allowed to be too high.

117. The examples of energy we have hitherto taken as illustrations are energies of systems, the motions and configurations of whose parts are manifest. Our grandest sources of energy are, however, derived from systems, the motions and configurations of whose parts are imperceptible. Whence the energy which enables the labourer to dig the ground, the student to pursue his studies, or the horse to draw his load? These are examples of *vital energy* which the man and horse derive from the food they eat and drink, and the air they breathe. The energy of gunpowder, of steam, and of a voltaic battery are other examples of what is called *molecular energy.*

118. Food and fuel are our principal immediate sources of energy. Thus coal and the oxygen of the air form a system which, before combustion, in virtue of the separation of the atoms of coal and the atoms of oxygen, possesses *potential energy of atomic separation.* During

combustion the energy becomes kinetic, and may be communicated to the water in a boiler so as to heat the water and form steam, and through this be used to drive an engine, and by means of the engine do all sorts of mechanical work. Similarly food and air form a great store of potential molecular energy, which is transformed during digestion into the vital energy by means of which we do our daily work. Winds and running water, including waterfalls, such as Niagara, and the ocean tides, are other considerable sources of energy made use of by man.

119. Heat, light, and electricity, in their physical aspects are well defined as forms of molecular energy. Sound forms a sort of connecting link between molar and molecular energy. The *Transformation of Energy* is the enunciation of the fact:

Any one form of energy may be transformed, directly or indirectly, into an equivalent of any other form.

120. Amongst the most important of the modern advances in Physical Science is the measurement of the different forms of molecular energy in dynamical units. Thus the energy of a *unit of heat*, (the heat required to raise the temperature of 1 gram of water from 4°C to 5°C), has been determined experimentally to be nearly equal to 42 million ergs. From such measurements the very important generalization, known as the *Conservation of Energy*, has been deduced:

Through whatever forms energy may pass, it cannot be changed in quantity, and hence the total energy in the universe remains constant.

As the *Conservation of Mass* forms the foundation of modern chemistry, the *Conservation of Energy* may be said to form the foundation of modern physics.

121. Although the total energy in the universe remains constant, it is gradually being transformed into lower forms so as to be less useful to man. This is the principle

enunciated by Lord Kelvin, and known as the *Dissipation* or *Degradation of Energy:*

The energy of the universe is gradually being transformed into a form in which it cannot be made use of by man, viz., that of uniformly diffused heat.

122. Perhaps the principal force through which energy is being constantly dissipated, or degraded into the useless form of diffused heat, is friction (art. 60). The direction of this force is always diametrically opposite to the direction of motion, or to that in which motion *would* take place under the influence of the other acting forces. When the surfaces between which friction is called into play are plane, and sliding motion does, or is just about to take place, the *law of friction*, determined by experiment to an approximation of the first degree, and sufficiently accurate for practical purposes, may be thus expressed:

For like surfaces the friction varies directly as the normal pressure between the surfaces, and is independent of the areas of the surfaces in contact, and of the relative speed between the surfaces: $F = kR$. Or thus:

For like surfaces the friction per unit of area depends only upon, and varies directly as, the normal pressure per unit of area between the surfaces: $f = kr$.

When there is no relative motion between the surfaces, the friction may have any value from 0 to the maximum value, which is reached when motion is just about to take place. The constant k which measures the ratio of the maximum friction to the normal pressure is called the *coefficient of friction* for the two surfaces in question. Rankine has shown that the value of k lies between 0·2 and 0·5 for wood on wood, 0·2 and 0·6 for wood on metals, 0·3 and 0·7 for metals on stone, and 0·15 and 0·25 for metals on metals.

On account of the dissipation of energy through friction and other causes, a machine does not do as much useful

work as the equivalent of the energy imparted to it. The ratio of the useful work done to the energy supplied is called the *efficiency* or *modulus* of the machine. The *duty* of a steam-engine is the amount of useful work performed per unit mass of fuel consumed.

Examination XII.

1. Define energy, work, and the configuration of a material system. How is work physically manifested?

2. Define kinetic and potential energy, and give three good examples of each.

3. How is work measured? Give examples. Name and define the unit of energy and work.

4. Determine the kinetic energy of a body whose mass is m, and speed v.

5. Prove that the potential energy of a body whose mass is m and height above the earth's surface h, is mgh.

6. Prove that when a body is moving vertically, under no other force than its weight, its total energy, relatively to the earth, is independent of its position.

7. Define a kilogrammetre and foot-pound, and determine their values in absolute measure.

8. If a body of m pounds be moving with a speed of v vels; find its kinetic energy in foot-pounds.

9. Name and define the unit of activity in absolute and gravitation measures, according to both the C.G.S. and F. P. S. systems.

10. Give various examples of molecular energy, both kinetic and potential.

11. What are our principal sources of energy?

12. Define the unit of heat, and give its measurement in ergs and kilogrammetres.

13. Enunciate the principles known as the Transformation, Conservation, and Degradation of Energy.

14. Enunciate the law of friction for plane surfaces in two ways, and define the coefficient of friction.

15. Define the modulus of a machine. and the duty of a steam-engine.

16. Prove that when a body is projected upwards in the atmosphere, the time of ascent is less than the time of descent.

Exercise XII.

1. How much work must be done to pump 1000 cub. ft. of water from a mine 150 fathoms deep?

2. In pile-driving 30 men raised a rammer of 500 kilograms through a height of 40 metres 12 times in an hour; find the average rate of working per man?

3. How many ergs of potential energy are there in a mill-pond near Kingston, Ont., which is 40 m. long, 20 m. broad, and 1 m. deep, and has an average fall of 5 metres?

4. A ball of 40 lbs is moving at the rate of 300 miles per hour; find its kinetic energy in ft.-lbs.

5. A machine (modulus $\frac{2}{3}$) for raising coals is worked by two horses; how much coal will be raised in a day of 8 working hours from a pit 90 metres deep?

6. An engine is found to raise 6 tons of material per hour from a mine 110 fathoms deep; find the horse-power of the engine, supposing $\frac{1}{5}$ of its energy to be lost in unavoidable resistances.

7. A railway train of 300 tons, in passing over a certain mile, has its speed increased from 40 to 50 miles per hour. If the average friction be 10 lbs.-wt. per ton, find the work done by the engine in passing over the mile.

8. What must be the horse-power of an engine whose modulus is $\frac{4}{5}$, working 8 hours per day, which supplies 3000 families with 100 gallons of water each per day, the mean height to which the water is raised being 60 feet?

9. How many bricks will a labourer raise to the mean height of 20 ft., working 8 hours per day; given that the mass of 17 bricks is 125 lbs., and that the average rate of doing such work is 1200 ft.-lbs. per minute?

10. If a load be 10 bricks (Ex. 9), and the man's own mass 140 lbs., what is the rate at which he expends his vital energy when working?

11. What would be the cost per ton to raise coals from a pit 25 fathoms deep, allowing \$3 per day for a horse and driver, and that the horse performs 24000 ft-lbs. of work per minute, working 8 hours per day?

12. At what rate will a train of 100 tons be drawn by a locomotive-engine of 70 H.P., the frictional resistance being 10 lbs.wt.-per ton; and how far, after steam is shut off, will it go before being brought to rest?

13. If 8 lbs-wt. per ton (Ex. 12) be the average frictional resistance until full speed is attained, how long will it take for the train to attain its maximum speed after starting; and how far will it have travelled in this time?

14. In what time will a locomotive of 100 force-de-cheval, drawing a train of 100 tonnes, complete a journey of 100 kilometres, supposing that the frictional resistance until full speed is attained, and after steam is shut off until it stops, be on the average 3 kilograms-weight per tonne, and after full speed is attained. 4 kilograms-weight per tonne. (1 tonne $=10^6$ grams.)

15. Determine the H. P. of the river Niagara which has a total descent of 334 feet, and discharges about 4×10^7 tons of water per hour.

16. A body rest on a rough horizontal board, which is moving horizontally; determine the maximum acceleration the board can have without the body slipping.

17. Shew that it requires as much work to increase the speed of a ship from 24 to 25 miles per hour as to give it the first 7 miles per hour.

18. A ball of 10 kilograms is fired from the mouth of a cannon 3 metres long with the speed of 15 kilotachs: find the mean pressure of the gaseous products on the ball.

19. There were 4000 cub. ft. of water in a mine of depth 60 fathoms, when an engine of 70 H.P. began to work the pump; the engine worked for 5 hours before the mine was cleared of the water; if the modulus of the engine were $\frac{2}{3}$, find the rate at which water was entering the mine.

20. Find in dyntachs the rate at which a fire-engine works, which discharges 10 kilograms of water per second with a speed of 1500 tachs.

21. A railway carriage of 5 tons mass is started on a level railroad with a speed of 8 vels, and moves over 200 ft. before it stops; determine the coefficient of frictional resistance.

22. A cistern is 10 ft. long, 7 ft. broad. and 8 ft. deep. The height of the top of the cistern from the water in the well is 56 ft. If a man can work with a pump at the rate of 2600 ft.-lbs. per minute, and the modulus of the pump is 0·66, how long will he take to fill the cistern?

23. A spring tide raises the level of the river Thames, between London and Battersea bridges, on an average 15 feet. If 5 miles be the distance between the bridges and 900 feet the mean breadth of the river, find the potential energy of the spring tide when full.

Answers.

1. $5{\cdot}616\times10^7$ ft.-lbs. 2. 20/9 kilogrammetres per sec.
3. $3{\cdot}622\times10^{14}$. 4. 120,373. 5. 32 tonnes.
6. 5. 7. 16,948 ft.-tons. 8. 14·2. 9. 3916·8.
10. 3484·8 ft.-lbs per min. 11. $7\frac{13}{16}$ cents.
12. $26\frac{1}{4}$ miles per hr.; 4608 ft.
13. 3 min. $19\frac{1}{2}$ sec.; 1280 yds. 14. 1 hr. 37 min. 23·3 sec.
15. $13\frac{1}{2}$ millions nearly. 16. kg. 18. $3{\cdot}75\times10^9$.
19. 55·22 cub. ft. per min. 20. $1{\cdot}125\times10^{10}$.
21. 1/201 nearly. 22. $20\frac{4}{11}$ hrs.; 1668×10^8 ft.-lbs.

Chapter XIII.

Action and Reaction.

123. *Two heavy bodies are connected by an inextensible string which passes over a fixed smooth peg, (or pully, as in Attwood's machine); required to determine the tension of the string.*

Let T denote the tension of the string, m and m' the masses of the bodies, m being the greater. Since the tension of the string is the same throughout, if the weight of the string may be neglected, by Newton's third law (art. 58); the acceleration of the heavier body will be $(mg - T) \div m$ downwards, and of the lighter body $(T - m'g) \div m'$ upwards; since these must be equal,

$$g - \frac{T}{m} = \frac{T}{m'} - g, \quad \therefore \quad T = \frac{2mm'}{m+m'} g$$

Cor. The acceleration $= \dfrac{mg - T}{m} = \dfrac{T - m'g}{m'} = \dfrac{m - m'}{m+m'} g$ as already proved (art. 68). If $m = m'$, the tension of the string is mg, and there is no acceleration, so that the bodies must either be at rest or moving with uniform speed.

The above completes the solution of the problem of Attwood's machine (art. 68), when the weight and mass of the string, the pully's mass, and friction may be neglected.

124. As an additional illustration of Newton's third law let us consider one of the very simplest cases of impulse (art. 54), viz., the direct impact of two spherical particles. If the centres of two spheres move in the straight line joining them, and the spheres impinge on one another, the impact is called direct; otherwise, the impact is called oblique.

Denote by m_1, m_2, the masses of the particles, and by u_1, u_2, their velocities before impact. If the direction of u_1 be called $+$, u_2, will be $+$ or $-$ according as m_2 is or-

iginally moving in the same or opposite direction to m_1. The action which takes place during impact may be explained thus:

a). Alterations of form and volume take place by work being done against the molecular forces, until the relative velocity of the two bodies is destroyed. If R denote the stress during this first stage of the impact, and v the common velocity, we get from Newton's dynamical laws,

$$R = m_1(u_1 - v) = m_2(v - u_2) \quad \dots \qquad (a)$$

whence $v = \dfrac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \quad \dots \qquad (1)$

$$R = \frac{m_1 m_2}{m_1 + m_2}(u_1 - u_2) \quad \dots \qquad (2)$$

These equations contain the complete solution of the problem, if the bodies do not separate again after impact. This will be the case, when the force of *adhesion* between the bodies counterbalances the force of *elasticity*, which tends to separate them.

b). If the bodies be sufficiently elastic, they have the common velocity v only for an instant, for an amount of *molecular potential energy* has been stored up in consequence of the change of configuration of each sphere, and in the transformation of this energy into the kinetic form through the force of elasticity, the original forms and volumes are as much as possible restored. During this second stage of the impact it is evident that the bodies receive accelerations of momentum in the same directions as during the first stage, and if R' denote the stress called into play, and v_1, v_2 the velocities after impact,

$$R' = m_1(v - v_1) = m_2(v_2 - v) \quad \dots \qquad (b)$$

Now it has been proved by experiment, that if the impact do not make any sensible permanent alteration of form, the relative velocity of the bodies after impact bears a constant ratio to the relative velocity before impact, *i.e.*

$$v_1 - v_2 = -e(u_1 - u_2) \quad \dots \qquad (c)$$

where e is a proper fraction, whose value depends only upon the material natures of the spheres. From (a), (b), and (c), by algebraical analysis, $R' = eR$. Also

$$v_1 = u_1 - \frac{m_2}{m_1+m_2}(1+e)(u_1-u_2) \quad \ldots\ldots \quad \ldots\ldots \quad (3)$$

$$v_2 = u_2 + \frac{m_1}{m_1+m_2}(1+e)(u_1-u_2) \quad \ldots\ldots \quad \ldots\ldots \quad (4)$$

$$R+R' = \frac{m_1 m_2(1+e)}{m_1+m_2}(u_1+u_2) \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad (5)$$

The value of e was found by Newton to be $\frac{5}{9}$ for balls of compressed wool and steel, $\frac{8}{9}$ for balls of ivory, and $\frac{15}{16}$ for balls of glass. It is called by most writers the *coefficient of elasticity*, a name strongly objected to by Tait and Thomson, who call it the *coefficient of restitution.*

Cor. 1. If $m_2 = \infty$, and $u_2 = 0$, the case is that of a sphere impinging normally on a fixed plane. The equations (3), (4), (5), become then

$$v_1 = -eu_1\ ,\ v_2 = 0\ ,\ R+R' = m_1(1+e)u_1.$$

Cor. 2. If $m_1 = m_2$, and $e = 1$, then $v_1 = u_2$, and $v_2 = u_1$, *i.e.* the bodies interchange velocities. This may be shown to be nearly the case for balls of ivory or glass. Also, if $u_2 = 0$, and $m_1 = em_2$, then $v_1 = 0$, and $v_2 = eu_1$.

125. The following results are at once deduced from the preceding investigation:

1. *Whether the bodies be elastic or not, the total momentum is not affected by the impact.* (Art. 59.)

From (1), $\qquad (m_1+m_2)v = m_1u_1 + m_2u_2$

From (3) and (4), $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$

2. *The total molar kinetic energy after impact is less than before impact.*

$$\tfrac{1}{2}(m_1+m_2)v^2 = \tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 - \tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2\ ,$$

and $\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 =$

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 - \tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2.$$

What becomes of the molar kinetic energy lost? It is transformed into the *molecular kinetic energy of heat*, so that the bodies after impact are warmer than before impact.

126. The *Conservation of Momentum* (art. 59) teaches that change of momentum in a body or system of bodies must be produced by forces *external* to the body or system. Let any forces act upon a body of mass m and produce in it an acceleration a, then ma is the measure of the single force which would produce the same dynamical effect on the body. If, after Newton, we call a force measured by $-ma$ the *resistance to acceleration*, which the body offers in virtue of its mass and inertia, then *D'Alembert's Principle* at once follows as a corollary to Newton's third law:

The external forces acting upon a body (or system of bodies), together with the resistance (or resistances) to acceleration, form a system of forces in equilibrium.

This principle evidently amounts to saying that the molecular or internal forces acting within a body or system of bodies are themselves in equilibrium.

127. Newton published his axioms or laws of motion in his celebrated work "Philosophiae Naturalis Principia Mathematica." At the end of the scholium appended to his laws he points out that another meaning may be attached to the words action and reaction besides that of force:

If the action of an agent be measured by the product of its force into its velocity; and if, similarly, the reaction of the resistance be measured by the velocities of its several parts into their several forces, whether these arise from friction, cohesion, weight, or acceleration; action and reaction, in all combinations of machines, will be equal and opposite.

As pointed out by Tait and Thomson, this remarkable passage contains in it the foundation of that great modern generalization, the Conservation of Energy.

Examination XIII.

1. A string passing over a smooth peg connects two heavy bodies; determine its tension, 1) when the bodies have different weights, 2) when the weights are the same.

2. Two spherical particles impinge directly; describe the nature of the impact, and determine the equations of motion. Define stress.

3. What is denoted by e in the theory of impact? How can it be experimentally determined? Give its value for a few substances.

4. How do we deduce the equations of impact of a sphere on a fixed plane. Give the equations.

5. Determine under what conditions will two spheres, impinging directly, interchange velocities?

6. Prove that the momentum of a system of spherical particles is not altered by direct impacts of its component parts.

7. Determine the change of molar kinetic energy in both stages of impact, when two spheres impinge directly. What becomes of it?

8. Enunciate and explain D'Alembert's principle.

9. How can it be said that Newton in his third law laid the foundation of the science of energy?

Exercise XIII.

1. A boulder of 2 tonnes is rolled from the summit of El Capitan in the Yosemite valley, a rock rising vertically 3000 feet; find the speed of, and distance travelled by, the earth when the boulder strikes the ground. (See Ex. V, 6.)

2. A ball is let fall from a height h above a fixed smooth table, and rebounds to a height h', prove that e for the ball and table $= \sqrt{(h'/h)}$.

3. Find the tensions of the strings in Ex. VII, 5, 9, 15.

4. A chain 20 ft. long and mass $2\frac{1}{2}$ lbs. per ft. is hanging vertically, and is connected by a fine wire of insignificant mass, which passes over a smooth pully to a body of 56 lbs.; find the tensions 1) of the wire, 2) of the chain at its middle point, 3) of the chain 2 ft. from the free end.

5. Prove that in Attwood's machine, if the total mass of the moving bodies be constant, the greater the tension of the string is, the less is the acceleration.

6. A body of 5 kilograms, moving with a speed of 3 kilotachs, impinges on a body of 3 kilograms moving with a speed of 1 kilotach; $e = \frac{2}{3}$, find the speeds after impact.

7. Two bodies of unequal masses, moving in opposite directions with momenta equal in magnitude, meet; shew that the momenta are equal in magnitude after impact.

8. The largest gun in the United States in 1891, with a charge of 440 lbs. of prism powder, sent a projectile of 1000 lbs. with a speed of 1865 vels; if the mass of the gun and carriage were 100 tons, find the speed of recoil of the gun, and the potential energy in a lb. of powder.

9. The result of an impact between two bodies moving with equal speeds in opposite directions, is that one of them turns back with its original speed, and the other follows it with half that speed; find e and the ratio of the masses.

10. A bomb-shell moving with a speed of 50 vels bursts into two parts whose masses are 70 and 40 lbs. After bursting, the larger part turns back with a speed of 10 vels; find the speed of the smaller part.

11. A and B are two uniform spheres of the same material and of given masses. If A impinges directly upon a third sphere C at rest, and then C, on B at rest, find the mass of C in order that the velocity of B may be the greatest possible for a given initial velocity of A.

12. Find the necessary and sufficient condition that one body moves after direct impact with the original velocity of the other.

13. Two balls, each $\frac{1}{2}$ cub. decim., one of elm and the other of silver, are connected by an inextensible cord and immersed in Lake Ontario; find the tension of the cord in grs.-wt., and the acceleration, neglecting friction and the cord's weight.

14. Two particles of 1 and 2 kilograms are connected by a cord which passes over a smooth pully; this pully and a particle of 3 kilograms are connected by another cord which passes over a smooth fixed pully; neglecting the masses and weights of the pullies and cords, find the tensions of the cords and the accelerations of the three particles.

15. A jet of water is projected against an embankment so as to strike it normally. If the speed of the jet be 2500 tachs, and 50 kilograms of water strike the embankment per second, find the pressure of water against the embankment, 1) when the water does not rebound, 2) when it rebounds with a speed of 500 tachs.

16. *A* strikes *B* which is at rest, and after impact rebounds with a speed equal to that of B; shew that *B*'s mass is at least 3 times *A*'s mass.

17. If the sum of the masses of two impinging spherical masses be $2m$, find the greatest loss of molar kinetic energy for given values of e, u_1, and u_2.

ANSWERS.

1. $1{,}376\times10^{-13}$ cm. per year; $29{,}776\times10^{-21}$ cm.
3. 12 kilogrs.-wt.; 7·5 or 12·5 lbs.-wt.; 2.4 lbs.-wt.
4. 52·8, 26·4, 5·28 lbs.-wt. 6. 1750, $3083\frac{1}{3}$.
8. 9·325; 123,491 ft.-lbs. 9. $\frac{1}{4}$; 1:4. 10. 155.
11. $C=\sqrt{(A\,B)}$. 12. Ratio of masses $e:1$.
13. $(0{\cdot}82)\,g$; $437\frac{1}{3}$ grs.-wt. nearly.
14. 24/17 and 48/17 kilogrs.-wt.; $7g/17$, $5g/17$, $g/17$.
15. 127,486 and 152,983 grs.-wt. 17. $\frac{1}{4}m(1-e^2)(u_1-u_2)^2$.

Chapter XIV.

Dimensional Equations.

128. In the previous pages the student has been introduced to two distinct scientific systems of units, called the C. G. S. and F. P. S. systems respectively. In both systems three independent or fundamental units are chosen, and from these all others are derived. It is not necessary that any three special units be taken as the fundamental ones. The three, however, which are most easily fixed upon; and with standards of which, comparisons are most easily and directly made, at all times and at all places; and in relation to which the derived units are most easily defined, and are of the simplest *dimensions*, in virtue of the established relations between the different units; are the units of length, mass, and time.

Dimensional equations are such as express in algebraical form the relations between dynamical units, and are used more particularly to express how a derived unit depends upon the fundamental units.

129. Whatever units of length, time, and speed be used, $V \propto L/T$; where V measures the speed of a body moving with constant speed, and L is the distance passed over by the body in the time T. Now if we take the unit of speed as that in which unit of length is passed over in unit of time, the relation is expressed thus, $V = L/T$. Hence if v, l, t, denote the units of speed, length, and time in a scientific system, $v = l/t$. This is called a dimensional equation. It tells us that the unit of speed depends upon the unit of length to the first power *directly*, and the unit of time to the first power *inversely*. Hence if the unit of length be increased or diminished n times, so will the unit of speed be increased or diminished n times; and if the unit of time be increased or diminished n times, the unit of speed will be diminished or increased n times.

Similarly if m, a, M, f, w, h, denote respectively the units of mass, acceleration, momentum, force, work, and activity, in a scientific system of units.

$$a=\frac{v}{t}=\frac{l}{t^2},\ M=mv=\frac{ml}{t},\ f=\frac{M}{t}=\frac{ml}{t^2},\ w=fl=\frac{ml^2}{t^2},$$

$$h=\frac{w}{t}=\frac{ml^2}{t^3}$$

Hence if the unit of length be increased or diminished x times, the unit of mass y times, and the unit of time z times, the unit of activity will thereby be increased or diminished x^2y/z^3 times.

If i, o, denote the units of angle and angular velocity, $i=\text{arc}/\text{radius}=l/l=l^o$, *i.e.* the unit of angle is independent of the fundamental units; and $o=i/t=t^{-1}$.

If A, p denote the units of area and pressure-intensity,

$$A=l^2, \text{ and } p=f/A=m/(lt^2)$$

If V, d denote the units of volume and density,

$$V=l^3, \text{ and } d=m/V=m/l^3.$$

130. In whatsoever way the dimensions of a derived unit be deduced, they must of necessity always be the same. Thus (art. 22) $o=v/r=(l/t)/l=t^{-1}$. Similarly (art. 112) the dimensions of energy and work are mv^2, *i.e.* $(ml^2)/t^2$ as above.

When an equation occurs in which different units are involved, it is evident that the dimensions of each term relatively to the fundamental units must be the same; otherwise, by simply changing the values of the fundamental units, the equation becomes untrue. For examples see articles 35, and 123 to 125.

131. An important use of dimensional equations is to facilitate the calculations of the numerical relations between the derived units of different systems, when the numerical relations between the fundamental units are known. Thus if l', m', t' denote the fundamental units in

the F. P. S. system, and p' the derived unit of pressure-intensity, and l, m, t, p the corresponding C. G. S. units,

$$p' = \frac{m'}{l't'^2}, \; p = \frac{m}{lt^2} \therefore \frac{p'}{p} = \frac{m'}{m} \cdot \frac{l}{l'} \cdot \left(\frac{t}{t'}\right)^2$$

$= 453{\cdot}593 \times 0{\cdot}0328087 = 14{\cdot}8818$ (see tables art. 132), *i.e.* 1 poundal per square foot $= 14{\cdot}8818$ barads.

Ex. Find the units of length, mass, and time in a scientific system in which a mile per hour is the unit of speed, a pound-weight the unit of force, and a foot-pound the unit of work.

Let L, M, T, denote the fundamental units:

$$\frac{L}{T} = \frac{5280}{3600} \cdot \frac{l'}{t'} = \frac{22}{15} \cdot \frac{l'}{t'} \quad \ldots. \quad \ldots. \quad \ldots. \quad (1)$$

$$\frac{ML}{T^2} = 32\tfrac{1}{6} \times \frac{m'l'}{t'^2} = \frac{193}{6} \cdot \frac{m'l'}{t'^2} \quad \ldots. \quad \ldots. \quad (2)$$

$$\frac{ML^2}{T^2} = \frac{193}{6} \cdot \frac{m'l'^2}{t'^2} \ldots \quad \ldots. \quad \ldots. \quad \ldots. \quad (3)$$

$$\therefore \; L = l' = 1 \text{ foot}, \; T = \frac{15}{22}t' = \frac{15}{22} \text{ second, and}$$

$$M = \frac{193}{6}\left(\frac{15}{22}\right)^2 . \, m' = 14\frac{923}{968} \text{pounds.}$$

132. The following tables give the numerical relations between the C. G. S., F. P. S., and a few other frequently occurring units. The numbers in the tables of length and mass give the results of the most accurate observations made in comparisons of the French and English standards. Those in the other tables are calculated from the dimensional equations of the units, as explained in last article. Each number is true to the last decimal place given, and the mantissae of the logarithms of the true ratios are added,

I. Length or Distance.

			Mantissae.
1 foot	=	30·4797 centimetres	4840111
1 mile (statute)	=	1609·33 metres	2066451
1 metre	=	3·28087 feet	5159889

II. Area or Surface.

1 square foot	=	929·014 sq. centimetres	9680222
1 acre	=	40·4678 ares	6071101
1 are	=	1076·41 square feet	0319778
1 square kilometre	=	247·110 acres	3928899

III. Volume, Bulk, or Capacity.

1 cubic foot	=	28·3161 litres	4520332
1 gallon	=	4·54102 "	6571531
1 litre	=	61·0254 cubic inches	7855105

IV. Angle.

1 right angle	=	1·5707963268 radian	1961199
1 radian	=	57·295779513 degrees	7581226

V. Mass.

1 ounce avoirdupois	=	28·34954 grams	4525461
1 pound "	=	453·5927 "	6566661
1 gram	=	15·43235 grains	1884321
1 kilogram	=	2·204621 lbs. avoirdupois	3433339

VI. Density.

1 gram per cub. cm.	=	62·4262 lbs. av. per cub. ft.	7953672

VII. Time.

1 day (mean solar)	=	86400 seconds	9365137
1 sidereal day	=	86164·1 "	9353264
1 mean sidereal month	=	27·321661 days	4365071
1 mean synodic "	=	29·530589 "	4702721
1 sidereal year	=	31558149·6 seconds	4991116
" "	=	365·2564 days	5625978
1 mean tropical year	=	365·2422 "	5625809

A *solar day* is the time in which the sun apparently revolves around the earth. A *sidereal day* is the time of the apparent rotation of the sphere of the heavens. A *sidereal month* is the time in which the moon makes a complete revolution in the sphere of the heavens amongst the fixed stars. A *synodic month* is the time between two consecutive full moons. A *sidereal year* is the time in which the sun apparently makes a complete revolution in the sphere of the heavens amongst the fixed stars. A *tropical year* is the time between two consecutive appearances of the sun on the *vernal equinox*, one of the points in which the *equinoctial* cuts the *ecliptic;* it governs the return of the seasons, and varies slowly through a maximum range of about a minute on each side of the mean value. The student will do well to satisfy himself that a *positive* rotation of the earth would produce an apparent *negative* rotation of the sphere of the heavens. and a *positive* revolution of the earth around the sun would produce an apparent *positive* revolution of the sun around the earth. It follows from this that the number of sidereal days in the sidereal year exceeds the number of mean solar days by unity; whence the relation between these days.

VIII. Speed.

		Mantissae.
1 vel	= 30·4797 tachs	4840111
1 mile per hr., or 22/15 vels	= 44·7036 "	6503425

IX. Momentum.

1 poundvel	= 13825·4 gramtachs	1406772

X. Force (taking $g = 980·5$).

1 poundal	= 13825·4 dynes	1406772
1 kilodyne	= 1·0199 gram-weight	0085524
1 grain-weight	= 63·5354 dynes	8030157

XI. Pressure-intensity.

1 poundal per sq. foot	= 14·8818 barads	1726550
1 mean atmosphere	= 1·0136 megabarad	0058495

XII. Work and Energy.

1 foot-poundal	= 421394 ergs	6246883
1 kilogrammetre	= 7·23307 foot-pounds	8593228

XIII. Activity.

1 foot-poundal per second	= 421394 dyntachs	6246883
1 force de cheval	= 7354·05 megadyntachs	8665266

Examination XIV.

1. What determines the choice of fundamental units?

2. Why is the French method of forming multiples and submultiples of units the best?

3. Define a dimensional equation. Write down the dimensional equations between angular velocity, momentum, energy, angle, pressure-intensity, and density, and the fundamental units.

4. Determine the ratios of the units of acceleration, angular velocity, density, and the gravitation units of the rates of doing work, in the F. P. S. and C. G. S. systems.

5. Define the following terms: mean solar day, sidereal day, sidereal month, synodic month, sidereal year, tropical year, equinox, equinoctial, ecliptic.

6. How is the ratio of the sidereal day to the mean solar day determined? Calculate the ratio.

7. Check all the ratios in tables VIII to XIII, art. 132.

Exercise XIV.

1. If a kilometre and hour be the units of length and time, what number will express the mean value of g?

2. If a metre, kilogram, and minute were the fundamental units, what number would express the mean atmospheric pressure at the sea-level (1·014 megabarad).

3. If a metre per second, a kilogram-weight, and a kilogrammetre were the units of speed, force, and work, find the units of length, mass, and time, and the number which expresses the pressure at a kilometre-depth of ocean (s.w. 1·027).

4. If a metre, kilogram, and 10^{-4} of a day be the fundamental units, find in dyntachs and barads the derived units of activity and hydrostatic pressure.

5. If g, a kilogram-weight, and a force-de-cheval be the units of acceleration, force, and activity, find the units of momentum and pressure-intensity.

6. The units of speed, acceleration, and force are 1 kilometre per hour, g, and the weight of a kilogram; find the units of length, mass, time, and density in terms of the C. G. S. units.

7. The relation between g, and the time of oscillation (t) and the length (l) of a pendulum is $t = \pi \sqrt{(l/g)}$. If 1 second be the unit of time, and the length of the second's pendulum at the latitude of 45°, where $g = 980{\cdot}5$ tachs per second, be 100 units of length, find the unit of length in centimetres, and the number which measures the mean value of g.

Answers.

1. $127{,}072{\cdot}8$. 2. 36504×10^4. 3. 1m.; 9805 grs.; 1 sec.; $1{\cdot}037 \times 10^6$. 4. $10^{13}/864^3$; $10^5/864^2$.
5. 7·5 megagramtachs; $980{\cdot}5^3 \times 75^4 \div 10^5$.
6. $10^6 \div (36^2 \times 980{\cdot}5)$; 10^3; $10^3 \div (36 \times 9805)$; $36^6 \times 980{\cdot}5^3 \div 10^{15}$. 7. 0·993454 cm.; 986·96.

CHAPTER XV.

Composition of Velocities.

133. *If a body have simultaneously two velocities, represented by lines drawn from a point, the resultant velocity will be represented by the diagonal, drawn from that point, of the parallelogram described on the two lines as adjacent sides.*

Let a body have a velocity along the line AX, represented by AB, and at the same time let the line AX move parallel to itself, the end A always keeping on AY, with a velocity represented by AC. It is evident that every point in AX, and also the body moving along AX, has this velocity represented by AC, and therefore the body under consideration has *simultaneously* velocities represented by AB and AC. It is required to prove that AD, the diagonal through A of the parallelogram $ABDC$, represents the resultant velocity of the body.

1). In *any time t* let the body move along AX through the distance AP, and in the same time suppose that AX in virtue of its motion moves parallel to itself through the distance AA', so that at the end of time t the position of AX is $A'X'$. In $A'X'$ take $A'P'$ equal to AP, then evidently P' is the position of the moving body at the end of time t. Now $AP : AA' = AB : AC$, $\therefore A'P' : AA' = CD : AC$, $\therefore A, D$, and P' must be in one straight line. Hence *in any time t* the body is found to be in AD or AD produced, and therefore the diagonal AD represents the *direction* of the resultant velocity.

2). Because the ratio $AP' : AP$ is equal to the constant ratio $AD : AB$, and since AP varies as t, therefore also AP' varies as t, *i.e.* the velocity along AD is *uniform*.

3). Because $AP' : AP : AA' = AD : AB : AC$, therefore AD represents the *resultant speed* on the same scale that AB and AC represent the component speeds in the directions of AX and AY.

The above important theorem is called the *Parallelogram of Velocities.* It may be lucidly illustrated by the motion of a boat which is propelled directly across a stream whilst carried down by the current. The triangle of velocities (art. 134) is another way of expressing the same truth.

Cor. If a body have simultaneously two velocities represented by lines AB, AC, and if E be the middle point of BC, the resultant velocity is in the direction of AE and is measured in magnitude by twice the length of AE.

The Triangle of Velocities.

134. *If a body have simultaneously two velocities represented by lines AB, BC, the line AC represents the resultant velocity.*

Cor. 1. If a body have simultaneously three velocities represented by the sides of a triangle *taken in order* (*e.g.* AB, BC, CA; or AC, CB, BA) the body will be at rest.

Cor. 2. If AB. AC represent the velocities of a body at the beginning and end of any interval, BC represents the *total acceleration* during that interval.

The Polygon of Velocities.

135. *If a body have simultaneously velocities represented by lines AB, BC, CD. . . .LM, MN, the line AN represents the resultant velocity.*

Cor. If a body have simultaneously velocities represented by the sides of any polygon taken in order, the body will be at rest.

The polygon of velocities is immediately deduced by repeated applications of the triangle of velocities. Note that this theorem is true whether the lines representing

the velocities be all in one plane or not. The following may be taken as an important particular case.

The Parallelepiped of Velocities.

136. *If a body have simultaneously three velocities represented by the edges of a parallelepiped which meet at a point, the diagonal of the parallelepiped through the point will represent the resultant velocity.*

137. Since acceleration is measured by the change of velocity per unit of time (art. 24), it is evident that there are propositions relating to the composition of simultaneous accelerations exactly similar to those of the preceding articles for velocities. Hence

1. *The Parallelogram and Parallelepiped of Accelerations.*

If a body have simultaneously two or three accelerations represented by lines drawn from a point, the resultant acceleration will be represented by the diagonal, drawn from that point, of the parallelogram or parallelepiped described on the lines as adjacent sides or edges.

2. *The Triangle and Polygon of Accelerations.*

If a body have simultaneously accelerations represented by the sides of any triangle or polygon taken in order, the resultant acceleration will be zero.

Relative Velocity.

138. By the term *relative velocity* we denote the velocity of one body with respect to or relatively to another body. All velocity is relative (art. 14).

Ex. 1. If a body A be at rest, and another body B is moving eastwards with a speed 12 ; or, if B be at rest, and A is moving westwards with a speed 12; or, if A is moving westwards with a speed 4, and B is moving eastwards with a speed 8; in all three cases the velocity of B relatively to A is 12 eastwards, and the velocity of A relatively to B is 12 westwards; for evidently B is separating from A eastwardly with a speed 12, and A is separating from B westwardly at the same rate.

Ex. 2. Let two persons A and B start from the same point with equal speeds v, and let P_1, P_2, P_3, ... be the positions in successive units of time of A who is travelling northwards, and Q_1, Q_2, Q_3. . . . the corresponding positions of B who is travelling eastwards. At the end of the first unit of time, A is at a distance $Q_1 P_1$ or $v\sqrt{2}$ in a N.W. direction from B; at the end of the second unit of time, the distance is $Q_2 P_2$ or $2v\sqrt{2}$ in a N.W. direction; at the end of the third unit of time, the distance is $Q_3 P_3$ or $3v\sqrt{2}$ in a N.W. direction; and so on. Hence we learn that A is moving relatively to B with a velocity $v\sqrt{2}$ in a N.W. direction. Similarly, the velocity of B with respect to A is $v\sqrt{2}$ in a S.E. direction.

From these examples it is evident that the velocities of two bodies with respect to one another are equal in magnitude and opposite in direction.

139. *Having given the velocities of two bodies, (i.e. with respect to the earth), to determine their velocities relatively to one another.*

Let two bodies P and Q have velocities represented by AB, AC respectively. Give to each of them a velocity represented by BA; P is then brought to rest, and Q has a velocity represented by BC (art. 134). Since the relative velocities of the two bodies cannot be altered by giving to each the same velocity, BC must represent the velocity of Q relatively to P. Similarly by giving to each a velocity represented by CA, it is manifest that CB represents P's velocity relatively to Q. Hence

If two particles have velocities represented by lines drawn from a point, the line joining the other extremities of the two lines represents the relative velocities of the particles. Or thus:

If two particles P and Q have velocities represented by lines AB, AC, then BC represents the velocity of Q relatively to P, and CB the velocity of P relatively to Q.

The direction of relative velocity is not necessarily the direction of relative position at any time. The latter depends upon the relative positions of the bodies at the beginning of motion as well as on their relative velocities.

Cor. If two particles have accelerations represented by lines drawn from a point, the line joining the other extremities of the two lines represents the relative accelerations of the particles.

140. If a body have simultaneously two velocities or accelerations denoted by v_1 and v_2, and if i denote the angle between the directions of v_1 and v_2, v the resultant velocity or acceleration, i_1 the angle between the directions of v and v_1, and i_2, the angle between the directions of v and v_2, the following formulæ are easily deduced:

$$v^2 = v_1^2 + v_2^2 + 2v_1 v_2 \cos i \quad \ldots \quad \ldots \quad \ldots (1)$$

$$\sin i_1 = \frac{v_2}{v} \sin i, \text{ and } \sin i_2 = \frac{v_1}{v} \sin i \quad \ldots \quad \ldots (2)$$

$$\tan i_1 = \frac{v_2 \sin i}{v_1 + v_2 \cos i}, \text{ and } \tan i_2 = \frac{v_1 \sin i}{v_2 + v_1 \cos i}. \quad \ldots (3)$$

Examination XV.

1. Enunciate and prove the parallelogram of velocities. Give three good physical illustrations thereof.

2. Enunciate the triangle, polygon, and parallelepiped of accelerations.

3. Given the velocities of two bodies, determine their relative velocities. Apply your result, and illustrate by a figure, when the bodies are moving in the same or in opposite directions.

4. Explain the directions of the trade winds, the anti-trade winds, and the Gulf stream.

5. Simplify the equations in art. 140 when $i=0$, $\frac{1}{2}\pi$, and π: also when $v_2 = v_1$.

6. Given the velocities and accelerations of two bodies, express their relative velocities and accelerations by means of algebraical equations similar to those in art. 140.

EXERCISE XV.

1. Prove that the resultant of two equal velocities bisects the angle between them; and conversely.

2. The resultant of two equal velocities is equal to either of them; find their inclination.

3. A river is 1·2 miles broad; a boat is rowed directly across at the rate of 3 miles per hour; the current is 2 miles per hour; how far does the boat travel in crossing, and how long does it take her to cross?

4. Find the resultant of velocities 2, 2, 2, 3, 3, 3, which a particle receives simultaneously in directions parallel to the sides of a regular hexagon taken in order.

5. Two persons start from the same place, the first an hour before the second; they travel along roads inclined to one another at an angle $\frac{1}{3}\pi$, each with a speed of 150 tachs; find their relative velocity, and their distance apart at the end of 4 hours from the starting of the first.

6. The speed of light is 3×10^{10} tachs, and that of the earth in its orbit 3 megatachs; what is the maximum angular displacement of a star owing to the *aberration of light.*

7. Two straight railroads cross each other; a train on each line is approaching the junction with constant speed; what is the necessary and sufficient condition that the trains collide?

8. Two equal circles in the same plane touch each other, and from the point of contact two persons move along the circumferences in opposite directions with equal speeds; shew that each will appear to the other to move in the circumference of a circle of double the diameter of the real circles of motion, the observer being in the circumference of the circle of apparent motion of the other. What will be the apparent motions, if the two persons start from the point of contact in the same direction?

9. Two trains, 200 and 150 ft. long respectively, are travelling on a double track railroad, with speeds of 20 and 25 miles per hour respectively. How long do they take to pass one another, 1) when going in the same direction, 2) when in opposite directions.

10. If a body have simultaneously velocities represented by $p.\ OA$ and $q.\ OB$, its resultant velocity is represented by $(p+q)\ OC$, where C is a point in AB, such that $p.\ AC=q.\ BC$.

11. A man can row a boat at 4 miles per hr. If the current of a river be 2 miles per hr., in what direction must he row relatively to either bank so as to cross 1) at right angles to the current, 2) in the shortest time?

12. A ship is steaming due east across a strong southward current. At the end of 4 hours the ship is found to have gone 40 miles 30° south of east. Find the current.

13. To a man walking at 2 miles per hour the rain appears to fall vertically; when he increases his speed to 4 miles per hour, it appears to meet him at an angle of $\frac{1}{4}\pi$; find the velocity of the rain.

14. The wind blows along a railroad, and two trains moving with equal speeds, have the aqueous cloud-track of the one double that of the other; find the ratio of the speed of each train to that of the wind.

ANSWERS.

2. $\frac{2}{3}\pi$. 3. 7614·9 ft.; 24 min.

4. 2, in the direction of the middle velocity 3.

5. 150, $\frac{1}{3}\pi$ with either road; 1,946,998. 6. 20″·6.

8. Each will seem to move from and to the other in a line perpendicular to the common tangent.

9. $47\frac{8}{11}$; $5\frac{10}{33}$. 11. 1) $\frac{1}{3}\pi$, 2) $\frac{1}{2}\pi$ to either bank.

12. 5 miles per hr. 13. $2\sqrt{2}$ miles per hr. at $\frac{1}{4}\pi$ to vertical. 14, 3:1.

Chapter XVI.

Composition of Forces.

A. *Forces whose lines of action meet one another.*

141. A single force which would produce the same effect as two or more forces is called their *resultant*. The term *component* is correlative to resultant. The student should familiarize himself with the term *line of action of a force* by imagining the force to act on a body either through a stretched inextensible cord or through a rigid straight rod, when the line of the cord or rod will be the line of action of the force, and the point at which the cord or rod meets the body may be said to be the point at which the force acts on the body. A *rigid body* is one whose configuration remains constant, whatever forces act upon it. In the dynamics of solids (*stereodynamics*) the bodies are assumed to be rigid. unless otherwise stated. This gives only a first approximation to the complete solution of problems, but it is an approximation sufficiently accurate for most practical purposes. We speak of the *tension* of a stretched cord, assuming thereby that the stress between every pair of contiguous particles is the same. If we may neglect the weight of the cord, this may be taken as an immediate result of Newton's third law (art. 58), and if the cord passes round any smooth surface, *e.g.* a peg, the tension remains unaltered, as any smooth surface can only exert force normal to itself, and this will not affect the stress along the cord, as will appear in what follows. A force is completely represented by a straight line, when the line is the actual line of action of the force, and its length measures the magnitude of the force. Any parallel line of equal length would represent the force in magnitude and direction.

142. *The resultant of any number of forces which have the same line of action is their algebraical sum.*

Let $f_1, f_2, -f_3, -f_4, f_5 \ldots$ denote any forces (in dynes or poundals) acting on a particle of mass m (in grams or pounds). If $a_1, a_2, -a_3, -a_4, a_5, \ldots$ denote the accelerations (in tachs or vels per second) which the forces acting separately would respectively produce, then when all the forces act simultaneously, the resultant acceleration will be $a_1+a_2-a_3-a_4+a_5 \ldots,$ and therefore the resultant force $m\ (a_1+a_2-a_3-a_4+a_5 \ldots,)$. If therefore f denote the resultant force, $f=m\ (a_1+a_2-a_3-a_4+a_5 \ldots) =ma_1+ma_2-ma_3-ma_4+ma_5 \ldots =f_1+f_2-f_3-f_4+f_5 \ldots$

143. From the parallelogram of velocities we passed at once to the parallelogram of accelerations (art. 137). When now we take into consideration the mass of the moving body. we at once deduce the parallelogram of accelerations of momentum, or as it is more commonly called,

The Parallelogram of Forces.

If two forces acting on a particle be represented in magnitude and direction by lines drawn from a point, the resultant force will be represented in magnitude and direction by the diagonal, drawn from that point, of the parallelogram described on the two lines as adjacent sides.

Denote by f_1, f_2, and m, the forces and the mass of the particle, and by f the resultant force. If a_1, a_2, a denote the accelerations produced by f_1, f_2, f respectively, then $a_1=f_1/m$, $a_2=f_2/m$, $a=f/m$, (art. 56). Let the lines AB, AC represent f_1, f_2, and complete the parallelogram $ABDC$. Since $a_1 : a_2=f_1 : f_2$, AB, AC may be taken to represent the accelerations a_1, a_2, and then AD will represent the resultant acceleration a (art. 137.) Therefore AD must be the *direction* of the resultant force f, and since $a_1 : a_2 : a =f_1 : f_2 : f$, AD must also represent the resultant force in *magnitude* on the same scale as AB, AC represent the components.

Cor. 1. If AB, AC represent two forces acting upon a particle A, and if E be the middle point of BC, the resultant will be completely represented by $2\,AE$.

Cor. 2. If two equal forces (f, f) act upon a particle at an inclination i, the resultant is $2f \cos \frac{1}{2} i$, and is equally inclined to the components.

Cor. 3. When f is substituted for v in art. 140, the formulæ apply equally well to forces acting upon a particle.

144. The parallelogram of forces can be proved experimentally by simple mechanical contrivances, as explained in treatises on Experimental Physics.

The triangle of forces is another way of stating the same truth. Observe the use of the term particle in these enunciations. If body were used, it would be necessary to add that the lines of action of the forces were concurrent.

The Triangle of Forces:

If three forces acting upon a particle can be represented in magnitude and direction by the sides of a triangle taken in order, they will keep the particle in equilibrium. Or thus:

If two forces acting on a particle be represented in magnitude and direction by two sides of a triangle taken in order, the resultant force will be represented by the third side taken in the reverse order.

Conversely: *If three forces acting on a particle keep it in equilibrium, and a triangle be drawn having its sides parallel to the lines of action of the forces, the magnitudes of the forces will be proportional to the lengths of the sides of the triangle respectively parallel to them.*

Cor. 1. If three forces keep a particle in equilibrium, and a triangle be drawn having its sides perpendicular to the lines of action of the forces, the magnitudes of the forces will be proportional to the lengths of the sides of the triangle respectively perpendicular to them.

Cor. 2. If P, Q, R represent three forces which keep a particle in equilibrium, and A, B, C denote the angles between the directions of Q and R, R and P, P and Q respectively, then $P:Q:R = \sin A : \sin B : \sin C$.

The Polygon of Forces:

145. *If any number of forces, whose lines of action are concurrent, can be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium.* Or thus:

If any number of forces, whose lines of action are concurrent, be represented in magnitude and direction by the sides of a polygon but one, taken in order, the remaining side taken in the reverse order will represent the resultant force in magnitude and direction.

The polygon of forces is immediately deduced by repeated applications of the triangle of forces. The converse is not true. This will be understood at once when it is remembered that equiangular polygons are not necessarily similar. Note further that the polygon of forces is true whether the forces all act in one plane or not. It contains a geometrical solution of the problem: *To determine the resultant of any number of forces acting upon a rigid body whose lines of action are concurrent.* The parallelepiped of forces may be taken as a particular case.

Resolution of Velocities and Forces.

146. Just as two or more velocities or forces can be compounded into one resultant, so can any velocity or force be resolved into two or more components. Thus if AD represent a velocity or force, and it be desired to resolve it into components in the directions of AX and AY, draw DB, DC parallel to AY and AX respectively, to meet these lines in B and D, then the components will be represented by AB and AC. When the components are to be in one plane, the solution is determinate if there be only two components, but indeterminate if more than two (art. 145).

The only important case of the resolution of force is that in directions at right angles to one another. The component in any direction is then called the resolved force, or the resolute, or the principal component in that direction, or better, simply *the component in that direction;* it measures the *effectiveness* of the force in that direction. If i denote the angle between the line of action of a force f and its principal component in any direction, the component will be measured by $f \cos i$.

147. *The algebraical sum of the principal components in any direction of two forces, which act upon a particle, is equal to the principal component in the same direction of the resultant of the two forces.*

Let AB, AC represent the forces, and AD their resultant. If XAY be the direction in which the forces are to be resolved, draw BE, CF, DG perpendicular to XAY. Since AC and BD are equal and parallel, their projections AF, EG on XY are equal. Now $AG = AE + EG = AE + AF$; which proves the proposition.

If AE be reckoned $+$, AF and AG will be $+$ or $-$ according as F and G lie on the same or opposite side of A as E does.

The proposition can evidently be extended to any number of forces whatsoever, and we therefore deduce: *The effectiveness in any direction of any number of forces acting upon a particle is measured by the principal component in that direction of the resultant of the forces.*

148. *To determine algebraically the resultant of any number of forces acting on a particle.*

For simplicity we shall confine ourselves to forces in one plane. Let O denote the particle, and $f_1, f_2, f_3, \ldots$ the forces. Draw through O any axes OX, OY, at right angles to one another, and let a_1, a_2, $a_3, \ldots$ denote the angles which the forces make with OX. Let R denote the resultant force, and A its inclination to OX. Then (art. 147)

$$R \cos A = f_1 \cos \alpha_1 + f_2 \cos \alpha_2 + f_3 \cos \alpha_3 + \ldots = \Sigma (f \cos \alpha)$$
$$R \sin A = f_1 \sin \alpha_1 + f_2 \sin \alpha_2 + f_3 \sin \alpha_3 + \ldots = \Sigma (f \sin \alpha)$$
$$\therefore\ R^2 = \{\Sigma(f \cos \alpha)\}^2 + \{\Sigma(f \sin \alpha)\}^2$$
$$\tan A = \Sigma(f \sin \alpha)/\Sigma(f \cos \alpha).$$

Cor. If the forces are in equilibrium, $R=0$; then
$$\Sigma(f \cos \alpha) = 0.\ \ \Sigma(f \sin \alpha) = 0$$
which express in algebraical language the necessary and sufficient conditions of equilibrium of any system of forces acting on a particle in one plane.

Examination XVI.

1. Define the terms resultant, rigid, the component in any direction, stereodynamics, equilibrium.

2. Enunciate and prove the parallelogram, triangle, and polygon of forces. Enunciate these conversely, and state which then remain true.

3. Deduce the algebraical equations between any two forces and their resultant, and write down the resultant of two equal forces P, P acting at an angle $2A$.

4. State and prove the relations between 3 forces in equilibrium, and the angles between the lines of action.

5. Determine, both geometrically and algebraically, the resultant of any number of forces acting upon a particle.

6. Prove that the algebraical sum of the components in any direction of any number of forces, which act upon a particle, is equal to the component in the same direction of the resultant of the forces.

7. Express, both geometrically and algebraically, the necessary and sufficient conditions of equilibrium of any number of forces, which act upon a particle.

Exercise XVI.

1. Prove that three equal forces, whose lines of action pass through one point, and are inclined to one another at an angle of $\frac{2}{3}\pi$, will be in equilibrium.

2. A cricket ball of 200 grams is moving eastward with a speed of 45 metres per second; find the impulse necessary to make it move northwards with an equal speed.

3. Prove that forces represented by lines drawn from the angular points of a triangle to the middle points of the opposite sides are in equilibrium.

4. A body of 10 kilograms is supported by two strings whose lengths are 1·2 and 0·9 metre; the other ends of the strings are fastened at two points in a horizontal line $1\frac{1}{2}$ metre apart; find the tensions of the strings.

5. Two forces acting at M are represented by MA and MB, and two others acting at N by NC and ND; shew that the four forces cannot be in equilibrium unless MN bisects both AB and CD.

6. Find the northward and eastward velocities of a ship which is sailing in a direction N. 30° E. with a speed of 12 miles per hour, and is carried in a S.W. direction by a current which flows at the rate of 2 miles per hour.

7. Shew that if the angle at which two given forces are inclined to one another be increased, their resultant is diminished.

8. The circumference of a circle is divided into any number of equal parts; shew that equal forces acting at the centre towards the points of division are in equilibrium.

9. A picture is suspended by a cord passed round a smooth pin and fastened to two rings on the picture frame; if 10 kilograms be the mass of the picture, and 30° the inclination of the two parts of the cord, find the tension of the cord.

10 A sphere of 2 tonnes rests on two smooth planes inclined to the horizon at angles of 60° and 30°; find the pressure on each plane.

11. Three pegs A, B, C are stuck in a vertical wall so as to form an equilateral triangle, having A highest and BC horizontal; a string passed over the pegs supports a kilogram at each end; find the pressure on each peg.

12. If the ends of the string (Ex. 11) be attached to a body of 2 kilograms, which is then supported at a point D, so that DBC is an equilateral triangle below ABC; find the pressures on the pegs.

13. A boat is tied by a rope to the right bank of a river flowing N.E., and is acted upon by a pressure S from the river and a pressure W from a S.E. wind; find the tension and direction of the rope.

14. A body of 12 lbs. is suspended from a point by a string 6 ft. long, and is acted on by a horizontal force of 9 lbs.-wt.; find how far the body is displaced and the tension of the string.

15. Two bodies of 5 kilograms each are connected by a string which is passed over 2 smooth pegs in a horizontal line 1 m. apart; a body of 3 kilograms is then hooked on to the string between the pegs; how far will it descend?

16. Forces are represented by 4 lines OA, OB, OC, OD; shew that their resultant is represented by 4 OG, G being the middle point of the line which joins the points of bisection of AC and BD.

17. Forces act at the middle points of the sides of a rigid polygon, in the plane of the polygon and at right angles to the sides; if the forces be proportional to the sides to which they are respectively perpendicular, shew that if they all act outwards or all act inwards, they will be in equilibrium.

18. A string is wrapped around a regular smooth polygon of n sides and pulled with a tension T; find the total crushing force at the angular points of the polygon. Hence determine the total pressure, and the pressure per unit of length, on the circumference of a smooth circular hoop in like circumstances.

19. The circumference of a circle (radius r) is divided into any number (n) of equal parts; find the resultant of a system of forces acting at one of the points of division and represented by the straight lines drawn from that point to the other points of division. See ex. 8.

20. Forces acting on a particle are represented by lines drawn to the angular points of a triangle from the centre of the circumscribing circle; prove that the resultant is represented by the line drawn from the same point to the intersection of the perpendiculars on the sides from the angular points.

21. Three forces P, Q, R represented by AB, AC, AD act on a particle A and keep it at rest; if the direction of P be fixed, but that of Q vary, find the locus of D which determines the direction of R.

Answers.

2. 1,272,792 gramtachs N.W. 4. 6 and 8 kilogrs-wt.
6. 8·9781 and 4·5858 miles per hr. 9. 5176·4 grs.-wt.
10. 1 tonne-wt.; 1732 kilogrs-wt. 11. 1732, 517·6, and 517·6 grs.-wt. 12. 2000, 1154·7, and 1154·7 grs.-wt.
13. $\sqrt{(S^2 + W^2)}$ at $\tan^{-1} W/S$ to the bank.
14. 3·8 ft.; 15 lbs.-wt. 15. 15·7.
18. $2nT \sin \pi/n$; $2\pi T$, T/r.
19. A force measured by nr towards the centre.
21. A sphere with radius equal to AC, and centre at E in BA produced so that $AE = AB$.

Chapter XVII.

Motion and Equilibrium on an Inclined Plane.

149. As practical applications of the principles of the preceding chapter let us consider (1) the motion of a heavy body sliding on an inclined plane, (2) the conditions of equilibrium of such a body.

A body slides down an inclined plane, to determine the nature of the motion.

The body is acted upon by two forces; (1) its weight or the attraction of the earth, which acts vertically downwards, (2) the pressure of the plane. The latter is generally divided into the two principal components, 1) the normal pressure of the plane, 2) the tangential action or friction, which being always opposite to the direction of motion, acts along the plane upwards. Let m denote the mass of the body, mg its weight (art. 63), R the normal pressure of the plane, F the friction, and a the inclination of the plane to the horizon. Resolving the forces along and perpendicularly to the plane the components are

$$mg \sin a - F, \text{ and } R - mg \cos a$$

Since there is no motion perpendicular to the plane

$$R - mg \cos a = 0, \therefore R = mg \cos a.$$

If k denote the coefficient of friction between the body and the plane, $F = kR = kmg \cos a$ (art. 122); therefore the resultant force acting along the plane which moves the body is $mg \sin a - k\,mg \cos a$. The acceleration of the body is therefore $g(\sin a - k \cos a)$ which is constant. From this we see that the motion of a body sliding down an inclined plane is exactly similar to that of a body falling freely, the only difference being that the acceleration of the body is less.

Cor. 1. If a body be projected *up* an inclined plane, the motion is uniformly accelerated. the acceleration being g (sin $a+k$ cos a) *down* the plane.

Cor 2. If the plane be *smooth*,the acceleration is g sin a downwards, whether the body be moving up or down.

150. If there be no acceleration, but the body either moving or *just about* to move, g (sin $a-k$ cos a)$=0$, and therefore $k=\tan a$.

a is then called the *angle of friction* or *angle of repose*. It is the greatest inclination a plane can be made to take without the body, when laid on the plane, actually sliding down. At any less inclination the body will not slide, but then the maximum friction is not called into play. The angle of repose is beautifully illustrated in the slopes of moving sand-dunes.

The above value of k points out one of the best and simplest methods of practically determining the coefficient of friction between two surfaces.

151. The following propositions are of interest in illustrating principles and affording intellectual exercise.

1. *The time of sliding from rest down any smooth chord of a sphere drawn from the highest or lowest point is constant.*

Let d denote the diameter, and i the inclination of the chord to the vertical diameter. The length of the chord will be d cos i, and the acceleration of any body sliding down the chord g cos i (art. 149, cor. 2), therefore the time of descent down the chord will be $\sqrt{}(2d/g.)$ This is a constant quantity, and is the time of falling freely through a diameter of the sphere, which might be expected as the vertical diameter is one of the chords.

2. *If two spheres touch at their highest or lowest points, the time of sliding from rest down any smooth straight line, intercepted between the surfaces and passing through the point of contact, is constant.*

Let A be the point of contact, AB and AC the vertical diameters of the spheres, and DE any line through A meeting the spheres in D and E. On BC as diameter, let a sphere be described, touching the other spheres in B and C. Join DB and EC. Let EC cut the sphere BC in F, and join BF. Evidently $DEFB$ is a rectangle, so that DE is parallel and equal to BF, and therefore the time of descent down DE is equal to that down BF, which is equal to the time of descent down BC (prop. 1), *i.e.* the time any body would take to fall freely through a distance equal to the difference or sum, according as the spheres touch internally or externally, of the diameters of the spheres.

152. In these propositions we have the keys to the solution of a set of interesting problems relating to *lines of quickest or slowest descent.* The following will serve as illustrations. The lines are supposed to be smooth.

1. *To find the lines of quickest and slowest descent from a given sphere to a given point without it.*

Let O be the centre of the sphere and A the point. Draw the vertical radius OB. Join AB and let C be the other point in which AB meets the sphere. CA will be the line of quickest or slowest descent according as B is the highest or lowest point of the sphere. Join OC and produce it to meet the vertical line through A in P. P is the centre of a sphere having A for its lowest point and touching the sphere O in C. Take any other point E in the sphere O. Join EA and let F be the other point in which EA meets the sphere P. Now the time of descent down CA is equal to the time of descent down FA (art. 151, 1), and therefore less or greater than the time of descent down EA.

2. *To find the lines of quickest and slowest descent from the higher of two given spheres, without one another, to the lower.*

Let O and P be the centres of the two spheres, A the highest or lowest point of the former, and B the lowest or highest point of the latter. Join AB and let C and D be the other points in which AB meets the spheres. CD is the line required.

Join PD and produce it to meet the vertical radius OA in Q. Q is the centre of a sphere which will touch the spheres O and P in A and D respectively. By last problem CD is the line of quickest or slowest descent from the sphere O to the point D. Take any other point E in the sphere P. Join AE and let F and G be the other points in which AE meets the spheres O and Q. FE is the line of quickest or slowest descent from the sphere O to the point E. But the time of descent down CD is equal to the time of descent down FG (art. 151, 2), and therefore less or greater than the time of descent down FE.

153. Let us now consider the conditions of equilibrium of a heavy body resting on an inclined plane and acted on by some force P in addition to those already considered.

Let a denote the inclination of the plane to the horizon, b the inclination of P's direction to the plane, W the body's weight, R the normal pressure of the plane, and k the coefficient of friction. P is supposed to act in the same vertical plane as W and R.

1. If P just prevents the body from sliding down, or the body is sliding down uniformly, the friction equals kR and acts along the plane *upwards*. Resolving the forces along and at right angles to the plane,

$$P \cos b + kR - W \sin a = 0, \quad P \sin b + R - W \cos a = 0,$$

whence $P = W \dfrac{\sin a - k \cos a}{\cos b - k \sin b}$, $\quad R = W \dfrac{\cos (a+b)}{\cos b - k \sin b}$

2. If the body is just about to move up the plane, or moves up uniformly, the friction will be along the plane *downwards*, and we get

$$P = W \frac{\sin a + k \cos a}{\cos b + k \sin b}, \quad R = W \frac{\cos (a+b)}{\cos b + k \sin b}$$

If f denote the angle of friction (art. 150), $k = \tan f$; substituting this value of k in the above, we find that for equilibrium P must lie between

$$W \frac{\sin (a-f)}{\cos (b+f)} \text{ and } W \frac{\sin (a+f)}{\cos (b-f)}$$

Cor. 1. If there be no friction $k=0$, and we get

$$P : W : R = \sin a : \cos b : \cos (a+b).$$

Cor. 2. If P act along the plane, b = 0, and we get

$$P : W : R = \sin a \mp k \cos a : 1 : \cos a.$$

Cor. 3. If the force P act along the plane and there be no friction, $k=0$ and $b=0$, and we get

$P : W : R = \sin a : 1 : \cos a$, *i.e.*

= height of plane : length : horizontal base.

Cor. 4. For different values of b, P in dragging up is least when $b=f$, it value being then $W \sin (a+f)$; and this becomes $W \sin f$ when $a=0$, *i.e.* when the body is dragged along a horizontal plane.

EXAMINATION XVII.

1. Write down the equations of motion of a body sliding down a rough inclined plane, and of a body projected up 1) a smooth, 2) a rough inclined plane.

2. A body is projected up a rough inclined plane, find the time taken to return to the point of projection, and the speed on reaching it. What becomes of the molar energy lost?

3. Define the angle of friction, and prove the relation between it and the coefficient of friction.

4. Shew that the kinetic energy acquired by a body sliding down a smooth plane is the same as it would have acquired in falling freely through the same vertical height.

5. Find the limiting values of the force which will keep a body in equilibrium on an inclined plane. Explain what takes place for other values of the force.

6. Find the direction and magnitude of the least force required to drag a body 1) up an inclined plane, 2) along a horizontal plane.

7. If a body is dragged up an inclined plane by a force acting along the plane, shew that the work done is the same as in dragging it along the base supposed to be of the same material as the plane itself, and then raising it vertically through the height of the plane.

Exercise XVII.

1. A body lies on a horizontal slab 10 feet long; if the coefficient of friction be $\frac{3}{4}$, how high may one end of the slab be raised before the body will begin to slide down.

2. Find the speed with which a body must be projected up a rough plane, inclined to the horizon at an angle of 30°, so as to travel just 10 metres up the plane, the coefficient of friction being $\tan 30^\circ$. Find also the time it will take to descend 10 metres, if projected downwards with a speed of 100 tachs.

3. A body of 10 kilograms hanging freely is connected by a cord passing over a small smooth pully with another body of 4 kilograms resting on a plane inclined to the horizon at $\frac{1}{3}\pi$; if $\frac{1}{2}$ be the coefficient of friction between the latter body and the plane, find the acceleration of motion and the tension of the cord.

4. A plane is inclined to the horizon at an angle $\frac{1}{6}\pi$; find into what two parts a body of 100 lbs. must be divided, so that one part, connected by a string with the other and hanging over the plane, may balance the other part resting on the plane, $k=\tan 30^\circ$.

5. At what rate can an engine of 30 horse-power draw a train of 50 tons up an incline 1 in 250, the resistance from friction being 7 lbs.-wt. per-ton?

6. It is found that it requires double the force acting along an inclined plane just to drag a body up, as it does just to keep the body from sliding down; find the relation between α and k.

7. A railway carriage. detached from a train when going up an incline of 1 in 280, is found to move over 1500 yds. before it begins to descend; if the friction be $6\frac{1}{4}$ lbs.-wt. per ton, find the speed of the train.

8. If the train (ex. 7) were going at the rate of 30 miles an hour on a level piece of road when the carriage was detached. how long and how far would the carriage move before stopping?

9. Find the lines of quickest and slowest descent from a point without a sphere to the sphere.

10. Find the lines of quickest descent between a sphere and a point within it.

11. Find the lines of quickest descent (1) from a straight line without a circle to the circle, (2) from a circle to a straight line without it, the circle and line being in the same vertical plane.

12. Find the lines of quickest descent between two spheres, one being within the other.

13. If l be the distance of a point from a plane and α the inclination of the plane to the horizon. find the shortest time in which a body can fall from the one to the other.

14. Find the locus of a point without a sphere of radius r, such that the shortest time in which a body can fall between the point and the sphere is equal to t.

15. Find the same (ex. 14) when t is the longest time a body can take to fall.

16. A body of 30 lbs. descending under the action of its weight draws another body of 30 lbs. up a plane 50 ft. long inclined at $\frac{1}{6}\pi$ to the horizon, by means of a cord passing over a small smooth pully; find when the cord must be cut,

in order that the ascending body may just reach the top of the plane, 1) when the plane is smooth, 2) when $k=\frac{1}{3}$.

17. Two bodies support one another on a rough double inclined plane by means of a fine string passing over the vertex, and no friction is called into play; shew that the plane may be tilted about either extremity of the base through an angle $2f$ without disturbing the equilibrium, f being the angle of friction and both angles of the plane being less than $\frac{1}{2}\pi - f$.

18. A body is kept in equilibrium on an inclined plane by a force in a given direction: prove that the pressure of the plane, if the plane be smooth, is an harmonic mean between the greatest and least normal pressures, if it be rough.

Answers.

1. 6 ft. 2. 1400·4; 10. 3. 387·75; 6045·8 grs.-wt.
4. The hanging body may be any mass not greater than 50. 5. 15 miles per hr. 6. $\tan\alpha=3k$.
7. 30 miles per hr. 8. 7 min. 17·7 sec.; 3210 yds.
9 and 10. The lines through the lowest and highest points of the sphere.
11. Through A the lowest or highest point of the circle draw the tangent AB meeting the line in B: take BC up or down the line equal to BA: join CA cutting the circle in D; CD is the required line.
12. The lines between the two lowest and the two highest points of the spheres. 13. $(\sec.\frac{1}{2}\alpha)\sqrt{(2l/g)}$.
14. Spheres of radius $r+\frac{1}{4}gt^2$, which touch the given sphere internally at its highest and lowest points.
15. Spheres of radius $\frac{1}{4}gt^2-r$, which touch the given sphere externally at its highest and lowest points.
16. After going 1) $33\frac{1}{3}$ ft., 2) 44·1 ft.

Chapter XVIII.

Composition of Forces.

B. *Forces whose lines of action are parallel.*

154. In Chapter XVI. we considered the composition of forces whose lines of action passed through one point, and could then neglect the dimensions of the body, or speak of the forces as acting on a particle. From the very nature of the case we cannot neglect the dimensions of a body in determining the resultant of parallel forces acting upon it. Forces whose lines of action are parallel and act in the same direction are called *like* parallel forces; if they act in opposite directions, they are called *unlike* parallel forces. A pair of like parallel forces may be illustrated by two men supporting a heavy bar, one man at each end. The vertically upward forces applied by the men balance the weight of the bar. A pair of unlike parallel forces is illustrated in breaking a nut by means of a pair of nut-crackers, the pressure of the nut being opposite to that of the hand on either arm of the crackers. Unlike parallel forces are conveniently distinguished by the signs + and −.

155. *To determine the magnitude, direction, and line of action of the resultant of two like parallel forces.*

Let P and Q denote the forces, and let A and B be their points of application. Since P and Q have the same direction, the direction of the resultant acceleration will be the same as that of the forces, and the magnitude equal to the sum of the accelerations produced by each force separately. Hence the resultant force R equals $P+Q$, and is parallel to P and Q. This may also be deduced from art. 143, cor. 3 by, making $i=0$.

To determine the line of action of R, we assume the following axiom: *The line of action of the resultant of*

any two concurrent forces passes through their point of intersection. If now a force S_1 act at A along AB, and an equal force S_2 at B along BA, R will evidently be the resultant of the four forces P, Q, S_1, S_2. Let X denote the resultant of P and S_1, and Y that of Q and S_2, and let the lines of action of X and Y intersect in D, then D must be a point in the line of action of R. If C be the point in which R cuts AB, CD is the line of action of R and is parallel to P and Q. To determine C, $\because P:S=CD:AC$, and $S:Q=BC:CD$ (art. 144), $\therefore P:Q=BC:AC$.

156. It is evident that the position of D depends upon the magnitude of S and the direction of P and Q, whilst that of C is independent of both. The position of C depends only upon the magnitudes of P and Q, and the positions of their points of application, and is hence called the *centre* of the two parallel forces.

157. *To find the magnitude, direction, and line of action of the resultant of two unlike parallel forces.*

We can deduce this directly as in art. 155, or thus:

Let P and Q denote the forces, Q being the greater, and A and B their points of application. Join AB and produce it to C, so that $AB:BC=Q-P:P$. If now a force $Q-P$ parallel and like to P act at C, the three forces P, Q, and $Q-P$ are in equilibrium (art. 155), and therefore the resultant of P and Q must be a force $Q-P$ parallel and like to Q, and having C for centre.

158. From arts. 155 and 157 it follows that if (P, Q) or $(P, -Q)$ be a pair of parallel forces acting at A and B respectively, and C be their centre, and if AB be denoted by $+l$ in magnitude and direction, then AC is denoted in magnitude and direction by $Ql/(P+Q)$ or $-Ql/(P-Q)$, and CB by $Pl/(P+Q)$ or $Pl/(P-Q)$.

159. By repeating the above processes it is evident that we can determine the magnitude and line of action of the resultant of any number of parallel forces whatsoever.

The magnitude is simply the algebraical sum of the components. The sign of this resultant indicates the direction: if it is $+$, the direction is the same as the $+$ components; if $-$, the same as the $-$ components. The resultant may be supposed to act at the *centre of the system.*

Def. The centre of a system of parallel forces is a point, fixed relatively to the points of application of the component forces, and through which the resultant of the system must pass, whatever be the direction of the component forces, provided their directions relatively to one another remain unchanged.

By repeated applications of arts. 155 to 157 the centre of any system can easily be found geometrically. In the following articles it is determined algebraically.

160. *Given the distances of the points of application of two like parallel forces from any plane, to determine the distance of their centre from the plane.*

Let two like parallel forces P and Q act at A and B, and let the distances AD and BE from a plane be denoted by p and q. Let C be the centre of P and Q, and R the resultant. Join AB, draw CF perpendicular to the plane, and through C draw a line parallel to DE to meet AD in G and BE in H. Denoting CF by r, we get

$P : Q = BC : AC$ (art. 155), $= BH : AG, = q - r : r - p$,

$\therefore r = (Pp + Qq)/(P + Q)$, or $Rr = Pp + Qq$.

161. *Given the distances of the points of application of two unlike parallel forces from any plane, to determine the distance of their centre from the plane.*

Let two unlike parallel forces P, and $-Q$ act at distances p, and $-q$ (A and B being on opposite sides of the plane in this case) from the plane. Suppose $Q > P$. Construct a figure as in last article. Then

$P : Q = BC : AC$ (art. 157), $= BH : AG, = r - q : p + r$,

$\therefore -r = (Pp + Qq)/(P - Q)$, or $R(-r) = Pp + Qq$

the distance of C from the plane in this case being $-r$.

Hence we get the following rule for determining the distance of the centre of any two parallel forces from a plane:

Multiply each force by the distance of its point of application from the plane, take the algebraical sum of the products, and divide by the algebraical sum of the forces.

162. *Given the distances of the points of application of force, in any system of parallel forces, from any plane, to determine the distance of the centre from the plane.*

Let the forces be denoted by $P, -Q, -R, S, T, \ldots$. and the distances of their points of application from the plane by $p, q, -r, -s, t, \ldots$.

By the previous articles the forces P, and $-Q$, are equal to a single force $P-Q$ acting at a distance $(Pp-Qq) \div (P-Q)$ from the plane; compound this with the parallel force $-$R, and we get a resultant $P-Q-R$ acting at a distance $(Pp-Qq+Rr)/(P-Q-R)$ from the plane; compound this with the parallel force S, and we get a resultant $P-Q-R+S$ acting at a distance $(Pp-Qq+Rr-Ss) \div (P-Q-R+S)$ from the plane. Thus we see that exactly the same rule to determine the centre of two parallel forces (art. 161) applies to any number of parallel forces. It may be concisely expressed thus: $d\Sigma(P)=\Sigma(Pp)$, where P denotes any force of a system of parallel forces, p the distance of its point of application from a plane, and d the distance of the centre of the system from the plane.

Examination XVIII.

1. Define unlike parallel forces; find directly the resultant of a pair of unlike parallel forces, and thence deduce the resultant of a pair of like parallel forces.

2. Shew how to find the centre of a pair of parallel forces 1) geometrically, 2) algebraically.

3. Define the centre of any system of parallel forces, and deduce the rule for finding its distance from any plane.

Exercise XVIII.

1. If two bodies balance each other on a straight lever in any one position inclined to the vertical, they will balance each other in any other position of the lever.

2. A shopkeeper uses a balance having arms 10 and 11 inches in length, and sells from the longer arm; what percentage of money drawn does he gain dishonestly?

3. Find the true mass of a body which balances a grams when placed in one scale of a false balance, and b grams when placed in the other; find also the ratio of the lengths of the arms, and how much the fulcrum should be shifted, $2l$ being the length of the beam.

4. A shopkeeper possessing a balance, whose arms are a foot and $12\frac{1}{2}$ inches long respectively, sells from each arm alternately; will he gain or lose in the long run? by how much p.c. of the money drawn?

5. If the arm of a cork-squeezer be 30 cm., and a cork be placed 5 cm. from the fulcrum. find the pressure on the cork, when 50 lbs.-wt. is applied by the hand.

6. Explain the boast of Archimedes, "Give me a lever and whereon to rest it and I shall move the world." What much easier way is there of moving the world?

7. A rod, whose weight may be neglected, rests between two pegs which are 1 ft. apart and in a horizontal line; a body of 10 lbs. is hung from one end of the rod, $1\frac{1}{2}$ ft. from the nearer peg; find the pressures on the pegs.

8. A man carries a bundle at the end of a stick over his shoulder; if the piece of stick between his hand and shoulder be shortened, is the pressure on the shoulder increased or diminished? Is his pressure on the ground altered thereby? Explain your answers.

9. Two bodies of P and Q lbs. balance at the ends of a lever whose weight is insignificant; if the bodies be interchanged, so that the greater P now hangs where Q was,

and Q where P was, find what additional weight must be added to Q to maintain equilibrium.

10. O is any point within a triangle ABC: like parallel forces act at A, B, and C, proportional to the areas BOC, COA, and AOB respectively; prove that O is the centre.

11. If O be outside of the triangle, and the forces in the same proportion as in ex. 10, under what condition may O be still the centre of the parallel forces?

12. Find a single body whose weight will produce the same effect as the weights of bodies of 1, 2, 3, 4, and 5 kilograms hanging on a rod at distances of 1, 2, 3, 4, and 5 decimetres from one end of the rod.

13. A square board (side 3 decim.) is kept horizontal by an attached string, when bodies of 1, 2, 3, and 4 lbs. respectively hang at the corners. Find the point where the string is fastened to the board.

14. Parallel forces K, L, M, N act at E, F, G, H, and $K : L : M : N =$ area FGH : area GHE : area HEF : area EFG, shew that the centre is at the intersection of EG and FH.

15. Like parallel forces of 3, 5, 7, 5 lbs.-wt. act at the angular points A, B, C, D respectively of a quadilateral, taken in order; shew that parallel forces of F, $10-F$, $4+F$, $6-F$ lbs.-wt., where F may have any value, acting at the middle points of AB, BC, CD, DA respectively, have the same centre and resultant.

Answers.

2. $9\frac{1}{11}$. 3. $\sqrt{(ab)}$; $\sqrt{a} : \sqrt{b}$; $l(\sqrt{a}-\sqrt{b})/(\sqrt{a}+\sqrt{b})$.
4. Lose, $\frac{1}{12}$. 5. 300 lbs.-wt. 7. 25 and 15 lbs.-wt.
8. Increased; no. 9. $(P^2-Q^2)/Q$. 11. The force at the vertex of the double angle, within which O lies, must be unlike to the other two.
12. 15 kilogrs. at $3\frac{2}{3}$ decim. from the same end of the rod.
13. 21 cm. from 12, and 15 cm. from 23.

CHAPTER XIX.

Couples. Moments.

163. There is one case of a pair of parallel forces for which the foregoing articles fail to give a single resultant, viz., the case of a pair of equal unlike parallel forces. According to art. 157 the resultant would be a force of indefinitely small magnitude, having a line of action at an indefinitely great distance; the effect of which it would be impossible to foresee. We must, therefore, as in all cases in which reasoning from established principles fail us, appeal to *experiment* to ascertain what is the dynamical effect of such a pair of forces. The answer is: the production, not of *translation*, but of *rotation* of the affected body about an axis normal to the plane of the couple.

A pair of equal unlike parallel forces is called a *couple*. The distance between the lines of action of the forces is called the *arm* of the couple. The *moment* of a couple is measured by the product of the numbers which measure the magnitude of either force and the length of the arm, and is + or – according as the couple tends to produce + rotation (*i.e.* opposite to the apparent rotation of the sphere of the heavens, when looking southwards), or – rotation.

It can easily be proved that two unlike couples of equal moments, in the same or parallel planes, balance one another. Hence, as is also proved by experience, the dynamical action of a couple is measured by its moment.

164. *If three forces can be represented in magnitude and line of action by the sides of a triangle taken in order, they are equivalent to a couple, whose moment is measured by twice the area of the triangle. (The complete Triangle of Forces).*

Let S, T, R denote the forces acting along the sides BC, CA, AB of the triangle ABC. By the triangle of

forces (art. 144) the resultant of S and T is a force R, whose line of action passes through C, and is parallel to BA, and therefore the system of forces is equal to a couple RR whose arm is the distance of C from AB, and whose moment is therefore measured by twice the triangle ABC, since R is represented by AB.

165. The *moment of a force about a point* is measured by the product of the numbers which represent the magnitude of the force and the perpendicular on its line of action from the point, and is + or – according as the force tends to produce + or – rotation about the point.

The moment of a force about any point measures the effect of the force in producing rotation about the point.

This may be taken as an experimental fact illustrated in the use of a lever, or it is easily seen that the force is equivalent to a couple, whose moment is the same as that of the force about the point, and an equal parallel force through the point, which evidently cannot produce rotation about the point.

Cor. Just as a force can be resolved into a couple and an equal and parallel force in the plane of the couple, so a force and couple in the same plane can be compounded into a single force equal and parallel to the original force.

166. The moment of a force about any point will evidently be measured by twice the area of the triangle formed by drawing lines from the point to the extremities of the line representing the force. The distance of the point from the line of action of the force is called the *arm* of the force about the point.

When will the moment of a force about a point vanish? Either 1) when the force itself vanishes, or 2) when the arm vanishes, *i.e.* when the point lies in the line of action of the force. In either case there can evidently be no tendency to rotation about an axis through the point.

The algebraical sum of the moments of the forces forming a couple, about *any* point in the plane of the couple, is evidently equal to the moment of the couple.

167. *The algebraical sum of the moments of two coplanar forces about any point in their plane is equal to the moment of their resultant about the point.*

1) When the forces are not parallel. Let S and T denote the forces, R the resultant. Let AB, AC, AD represent these forces. If O be the point about which moments are taken, join OA, OB, OC, OD. Suppose O lies between AD produced and CD produced, then

$$\text{moment of } S : \text{moment of } R = \triangle\, OAB : -\triangle\, OAD$$
$$\text{moment of } T : \text{moment of } R = -\triangle\, OAC : -\triangle\, OAD$$
$$\therefore \text{mo. of } S + \text{mo. of } T : \text{mo. of } R = OAB - OAC : -OAD$$
$$\text{now } OAB = OCD + DAB = OCD + ACD = OAC - OAD$$
$$\therefore \text{moment of } S + \text{moment of } T = \text{moment of } R.$$

2) When the forces are parallel. Take S and T unlike forces, S the greater, and let R denote their resultant. Draw through O a line cutting the lines of action of the forces in A, B, and C. Suppose O lies between S and R,

$$\text{moment of } S : \text{moment of } R = -S.OA : +R.OC$$
$$\text{moment of } T : \text{moment of } R = +T.OB : +R.OC$$
$$\therefore \text{mo. of } S + \text{mo. of } T : \text{mo. of } R = T.OB - S.OA : R.OC$$
$$\text{now } T.OB - S.OA = T.CB - T.OC - S.CA + S.OC$$
$$= (S - T)\, OC = R.OC$$
$$\therefore \text{moment of } S + \text{moment of } T = \text{moment of } R.$$

The student will find it very instructive to verify the proposition for all possible positions of the point O.

Cor. 1. When any number of forces act upon a body in one plane, the moment of the resultant force (or couple), about any point in the plane, is the algebraical sum of the moments of the component forces about the same point.

Cor. 2. If, therefore, the forces be in equilibrium, the algebraical sum of the moments is zero; if the system be

equal to a single resultant, the sum depends upon the position of the point, and vanishes only when the point lies on the line of action of the resultant: if the system reduces to a couple, the sum is a constant quantity, but not zero, whatever be the position of the point.

Cor. 3. Conversely, if the algebraical sum of the moments of any number of forces acting in one plane, about three points not in a straight line, be zero for all three points, the forces are in equilibrium; if the sum be not of the same value for all three points, the forces have a single resultant; if the sum be of the same value but not zero for all three points, the forces are equivalent to a couple.

168. When a body can move in any manner whatever, it is said to be *free;* if its motion be restricted in any manner or by any condition, it is said to be *constrained.* We have already considered a case of motion of a constrained body in Chap. XVII. A oscillating pendulum, a sliding window, a swinging door, a ring moving on a retort stand will serve as other illustrations of constrained bodies.

If a body can only rotate about a fixed axis, and is acted upon by a system of forces whose lines of action are all at right angles to the axis, it is required to find the necessary and sufficient condition of equilibrium.

The body will be in equilibrium if the forces do not produce *rotation* about the axis. The effect of any one force to produce rotation being measured by the product of the force into the distance of its line of action from the axis, *i.e.* by the moment of the force about the axis, the necessary and sufficient condition of equilibrium is, that the algebraical sum of the moments of the forces about the axis vanish.

In the wheel and axle, toothed wheels, and other forms of the lever we have practical examples of bodies constrained in the manner just considered.

169. If the line of action of any force P is not at right angles to the axis, resolve P into two components, one parallel to the axis, and the other at right angles to the axis. The moment of the latter about the axis, will evidently measure the effect of P in producing rotation about the fixed axis. This effect will vanish, (1) when P vanishes, (2) when the line of action P is parallel to the axis, (3) when the line of action of P meets the axis.

The other component will produce motion parallel to the axis, and may be neglected if the body can only rotate; if, however, the body can also slide parallel to the axis, as a ring on a retort-stand or a screw in its nut, there cannot be equilibrium, unless 1) the sum of the moments about the axis, of the components at right angles to the axis, vanish, and 2) the sum of the components parallel to the axis vanish. If, as in a sliding window, rotation is impossible, equilibrium is established, if 2) alone is satisfied.

170. *When three forces keep a body in equilibrium, their lines of action must all lie in one plane, and must be all concurrent or all parallel.*

Let R, S, T denote the forces. Take A, B, and C points in the lines of action of R, S, and T, such that BC, CA, and AB are not parallel to the lines of action of R, S, and T respectively. Since the body is in equilibrium, we may suppose BC a fixed axis. The forces S and T whose lines of action meet this axis, can have no effect in producing rotation about it, nor therefore, since the body is in equilibrium, can R produce rotation about it. The line of action of R must therefore cut BC (art. 169), and must therefore lie in the plane ABC. Similarly it may be shewn that the lines of action of S and T lie in the plane ABC.

If the lines of action be not all parallel, let two of them R and S meet in O, then the resultant of R and S passes through O, and, since this resultant is balanced by T, the line of action of T must also pass through O.

Examination XIX.

1. Define a couple, the arm and moment of a couple, and write down the dynamical dimensions of the moment of a couple or force.

2. Prove that two unlike couples of equal moments in the same plane balance one another, and shew that when three parallel forces are in equilibrium they are really a pair of such balancing couples.

3. Enunciate and prove the complete triangle of forces. Hence prove the complete polygon of forces: If a system of forces can be represented in magnitude and line of action by the sides of a plane polygon taken in order, it is equivalent to a couple whose moment is measured by twice the area of the polygon.

4. Define the moment of a force about a point, and prove from theory what it measures.

5. Find the magnitude and line of action of the resultant of a force P and a couple Qq in the same plane.

6. Prove that the algebraical sum of the moments of two coplanar forces about any point in their plane is equal to the moment of their resultant, and extend the proposition to any number of forces in one plane.

7. Any number of forces act upon a body in one plane; find the conditions that the forces can be reduced to 1) a single resultant, 2) a couple, 3) equilibrium.

8. Define the moment of a force about an axis: find the condition of equilibrium of a body which can only rotate about a fixed axis; and give examples of such bodies.

9. Prove that when three forces keep a body in equilibrium, their lines of action must all lie in one plane.

Exercise XIX.

1. If the sum of the moments of forces in one plane be of the same value, but not zero, for two points in the plane,

the straight line which join these two points is parallel to the resultant force, or the forces reduce to a couple.

2. In a wheel and axle the radii are as 8 to 3; two bodies of 6 and 15 lbs. are suspended from ropes wound round the wheel and the axle respectively; one is supported by a prop; find the pressures on the prop, and on the fixed supports of the wheel and axle.

3. Bodies of 1 and 4 lbs. are suspended from the ends of a straight lever of insignificant weight; the fulcrum and a point at which another body is suspended divide the lever into three equal parts; find the mass of the third body in order that the lever may be in equilibrium.

4. A lever of insignificant weight is 5 ft. long; two strings, 3 and 4 ft. long, attached to the extremities of the lever, support a body of 10 kilograms; if the lever be kept in equilibrium in a horizontal position, find the tensions of the strings and the position of the fulcrum.

5. Two coplanar forces S and T act at the ends of a straight lever AB, whose weight may be neglected; find the position of the fulcrum in order that there may be equilibrium, the inclinations of S and T to AB being a and b respectively; find also the pressure on the fulcrum.

6. Forces are represented by the perpendiculars drawn from the angular points of a triangle on to the opposite sides; find under what condition they are in equilibrium.

7. O is any point within a triangle ABC; AO, BO, CO cut the sides in D, E, F; find under what conditions forces represented by AD, BE, and CF are in equilibrium.

ANSWERS.

2. $\frac{3}{8}$ and $20\frac{5}{8}$ lbs.-wt. 3. 2 lbs.

4. 8 and 6 kilogrs.-wt.; $AC : CB = 9 : 16$.

5. $AC : CB = T \sin. b : S \sin. a$;

$$\sqrt{\{S^2 + T^2 - 2\,ST \cos\,(a+b)\}}.$$

6. That the triangle be equilateral. 7. D, E, F must be the middle points of the sides.

Chapter XX.

Centres of Weight and Mass.

171. We may consider any body or system of bodies to be made up of small particles, whose positions can be defined by geometrical points, and the weights of the particles may be supposed to act at these points, and to be parallel (art. 62). The centre of such a system of parallel weights is called the *centre of weight* or *centre of gravity* of the body or system of bodies. To save circumlocution in what follows, we shall use the term body either for a single body, or for a system of bodies whose configuration does not change.

It follows from art. 159 that the position of the centre of weight, relatively to the particles which compose the body, is constant, whatever be the position of the body relatively to the earth. Hence, *if the different parts of a body, acted on only by weight, be rigidly connected with the centre of weight, and if this point be supported, the body will balance in all positions.* This important property is sometimes used as a definition of the centre of weight. The following facts follow immediately from it:

1. If a body balances on a straight line (or axis) in all positions, the centre of weight must lie in that line.

2. If a body can turn freely round an axis which is not vertical, it cannot be at rest unless the centre of weight lies in the vertical plane through the axis.

3. If a body hang from a point round which it can turn freely, it cannot be at rest unless the centre of weight is in the vertical line through the point of suspension.

Hence when a body is suspended by strings attached to different points of the body, the lines of the strings, when the body is at rest, all pass through the

centre of weight. This gives a practical method of finding the centre of weight of any body, however irregular its configuration may be.

172. When a body is suspended by a *string*, the centre of weight will necessarily be below the point of suspension on account of the non-rigidity of the string. When it is *rigidly* connected with a point, the body can be supported at this point, provided the centre of weight and point of support be in the same vertical line. Should, however, the centre of weight be above the point of support, and any slight displacement take place, the weight of the body will cause it to rotate about the point, until the centre is below the point of support. If the centre of weight be below the point of support, and any slight displacement take place, the weight will bring the body back again to to its old position. If the centre of weight coincide with the point of support, and any slight displacement take place the body will remain displaced without any tendency either to recede further from, or to return to its former position. In these respective relative positions of the point of support and centre of weight, the body is said to be in *unstable, stable*, or *neutral equilibrium.*

173. *A body placed on a plane will stand or fall, according as the vertical line through its centre of weight passes within or without the base of support.*

By base in this statement is meant the polygon of greatest area which can be formed by joining points of contact of the body and plane.

174. Since weight is a vertically downward force, its effect on a body is to bring down the centre of weight as far as possible. Hence a body supported in any way from falling will be in stable equilibrium, *for a displacement in any direction*, if such a displacement raises the centre of weight; in unstable equilibrium, if the displacement lowers the centre of weight; and in neutral equilibrium,

if the displacement does not alter the vertical height of the centre of weight. A right circular cone of uniform density resting on a horizontal plane illustrates the three kinds of equilibrium according as it rests on its base, apex, or curved surface.

A body is *practically* in unstable equilibrium if it be in unstable equilibrium for a displacement in any direction whatsoever. A sphere of uniform density, resting on a saddle-back surface, presents the three kinds of equilibrium according to the direction of displacement, but practically it is in unstable equilibrium.

175. *Given the weights and Cartesian coordinates of the particles which compose a body, to determine the coordinates of the centre of weight.*

Take three axes OX, OY, OZ mutually at right angles to one another, and let x, y, z denote the coordinates of any one of the particles of the body having weight w, and a, b, c the coordinates of the c. of w., then (art. 162)
$a = \Sigma(wx) \div \Sigma(w)$, $b = \Sigma(wy) \div \Sigma(w)$, $c = \Sigma(wz) \div \Sigma(w)$.

Cor. When a number of bodies are raised through various heights, the work done is the same as that of raising a body, whose weight is the sum of the weights of the bodies, through the height that the c. of w. of the bodies is raised.

176. If in the above we write mg for w (art. 63) we get
$a = \Sigma(mx) \div \Sigma(m)$, $b = \Sigma(my) \div \Sigma(m)$, $c = \Sigma(mz) \div \Sigma(m)$, the coordinates of a point, which, although it coincides with the c. of w., is quite independent of weight. It is called the *centre of mass* or *centre of inertia* of the body. It may be also thus defined:

The *centre of mass* of a body is a point, which coincides with the centre of a system of parallel forces, acting at the particles which compose the body, and proportional to the masses of the particles.

For a body of uniform density the position of the c. of m. depends only upon its configuration, and is independent of its mass. The term *centroid* is then used for c. of m.; also centre of length, area, or volume according as the body is practically a line, surface, or solid.

Let it be observed that we can speak of the centre of mass of any body whatsoever, whether belonging to the earth or external to it, whilst, properly speaking, the term centre of weight is applicable only to bodies near the earth's surface, and whose dimensions are small.

177. *The centre of mass of a very thin straight rod of uniform density and section is its middle point.*

Cor. If the rod be thick, the centre of mass will lie in the middle section, for any such rod may be supposed to be made up of indefinitely thin rods.

178. *To find the centre of mass of a thin triangular plate of uniform density and thickness.*

Such a plate can be represented by a plane triangle ABC. Let D and E be the middle points of BC and CA. AD is the locus of the middle points of all lines parallel to BC. Now we may conceive the plate to be made up of an indefinitely large number of thin rods parallel to BC, and the centre of mass of each of these rods will lie in AD, therefore the centre of mass of the whole plate will lie in AD. Similarly it may be shewn that it will lie in BE. Therefore it is G the point of intersection of AD and BE. Since D and E are the middle points of BC and CA, DE is parallel to AB and equal to one half of AB; and the triangle DGE is similar to the triangle AGB; therefore DG is one-half of GA or one-third of DA.

Cor. 1. The medians of a triangle meet in the centroid of the triangle and trisect one another.

Cor. 2. The centre of mass of a thin triangular plate coincides with that of three equally massive particles

situated at the angular points of the triangle, or at the middle points of the sides.

Cor. 3. The centroid of a plate in the form of a parallelogram is the intersection of the diagonals.

Cor. 4. The centroid of a triangular prism is the middle point of the line joining the centroids of the opposite triangular faces.

Cor. 5. The centroid of a plate in the form of any plane rectilineal polygon may be determined by dividing the polygon into triangles and applying art 155.

179. *To determine the centroid of a thin plate in the form of any plane rectilineal quadrilateral.*

Let $ABCD$ represent the plate. Bisect BD in E; take EF one-third of AE, and EH one-third of CE; join FH, cutting BD in K; take HG equal to FK (or FG equal to HK); G is the centroid required.

180. *To find the centroid of a triangular pyramid.*

Let $ABCD$ represent the pyramid. Bisect CD in E; take EF one-third of BE, and EH one-third of AE; let AF and BH intersect in G; G is the centroid required. For we may imagine the pyramid to be made up of indefinitely thin triangular plates, all parallel to BCD. Let bcd represent one of these plates, cutting the plane $ABFE$ in bfe. Since dec is parallel to DEC, $de : DE, = Ae : AE, = ec : EC$; therefore $de = ec$. Again, because bfe is parallel to BFE, $ef : EF, = Af : AF, = bf : BF$; therefore ef is one-third of be, and f is the centroid of the plate bcd. Thus AF is the locus of the centroids of all the plates parallel to BCD, and therefore contains the centroid of the pyramid. Similarly it may be shewn that BH contains the centroid, and therefore G is the centroid required.

Because EF is one-third of EB, and EH one-third of EA, FH is parallel AB and is one-third of AB; and the triangle HFG is similar to the triangle ABG. Therefore FG is one-third of GA, or one-fourth of FA.

Cor. 1. The lines drawn from the vertices of a tetrahedron to the centroids of the opposite faces meet in the centroid of the tetrahedron, and quadrisect one another.

Cor. 2. The centre of mass of a triangular pyramid of uniform density coincides with that of four equally massive particles placed at the vertices of the pyramid.

Cor. 3. A pyramid whose base is any plane rectilineal polygon can easily be divided into triangular pyramids. The centroid of the pyramid will lie in the line joining the vertex with the centroid of the base, and be three-quarters of this line from the vertex.

Cor. 4. A cone may be considered to be a pyramid having a base with an indefinitely large number of sides, and therefore the rule for finding the centroid of a pyramid (Cor. 3) applies to a cone having any plane base.

181. A body is *symmetrical* with respect to a point, line, or plane, when the body may be conceived to be made up of pairs of equally massive particles, the two which form a pair being on opposite sides of the point, line, or plane, equidistant from it, and in the same perpendicular to it. The point, line, or plane will contain the centre of mass of every pair of particles, and therefore also the centre of mass of the whole body. From this principle of symmetry we can frequently find the centre of mass with great facility. Thus, the centroids of a circular or elliptic ring, of a circular or elliptic plate, of a sphere, spheroid, or cuboid are apparent.

182. *Having given the speeds of any number of particles in any direction, to determine the speed of their centre of mass in the same direction.*

Let m denote the mass of any one of the particles, and d its distance from a fixed plane at right angles to the direction in question, at any instant; then the distance of the centre of mass from the plane at the same instant will be $\Sigma(md) \div \Sigma(m)$. Let v denote the speed of the particle

m in the direction in question, then at the end of time t the distance of m from the plane will be $d+vt$, and therefore the distance of the centre of mass from the plane, at the end of time t, will be $\Sigma\{m(d+vt)\}\div\Sigma(m)$, that is $\{\Sigma(md)\div\Sigma(m)\}+\{\Sigma(mv)\div\Sigma(m)\}t$ which shews that the speed of the centre of mass is $\Sigma(mv)\div\Sigma(m)$.

If the speeds of the particles in the given direction be not all constant, t must be taken indefinitely small.

183. From arts. 59 and 182 we deduce the important fact, that the velocity of the centre of mass of any system of bodies cannot be altered by the mutual actions (*e.g.* direct impact, art. 125) of its several parts. Hence *the centre of mass of the universe, or of any body not acted on by external force, is either at rest or in uniform motion.*

184. *Having given the accelerations of any number of particles in any direction, to determine the acceleration of the centre of mass in the same direction.*

Let v denote the speed, in the given direction, of any particle having mass m, at any instant, then $\Sigma(mv)\div\Sigma(m)$ is the speed of the centre of mass at the same instant (art. 182). If a denote the acceleration of m, then at the end of time t the speed of m will be $v+at$, and therefore the speed of the centre of mass will be $\Sigma\{m(v+at)\}\div\Sigma(m)$, or $\{\Sigma(mv)\div\Sigma(m)\}+\{\Sigma(ma)\div\Sigma(m)\}t$, which proves that the acceleration of the centre of mass is $\Sigma(ma)\div\Sigma(m)$.

If the accelerations of the particles in the given direction be not all constant, t must be taken indefinitely small.

185. Articles 182 and 184 shew us, that at any instant, the total momentum and acceleration of momentum of any system of particles, is the same as that of the total mass of the system concentrated at the centre of mass. Hence, when any forces act upon a system of bodies, the motion of the centre of mass is the same as that of a particle, whose mass is the total mass of the system, and which is acted upon by the same forces. Hence, so far as *translation* is

concerned, the motion of any body is represented by the motion of its centre of mass.

186. *The kinetic energy of any system of particles is equal to the kinetic energy of the whole mass of the system moving with the speed of the centre of mass, together with the kinetic energies of the different parts of the system relatively to the centre of mass.*

Let OI represent the velocity of the c. of m. and IP that of any particle (mass m) of the system relatively to the c. of m., then OP represents the velocity of m. Draw PQ at right angles to OI; then $\Sigma(\text{m. IQ})=0$, (art. 182). Now $OP^2=OI^2+IP^2\pm 2\ OI.\ IQ$ $\therefore$ $\Sigma(\frac{1}{2}mOP^2)=\frac{1}{2}\Sigma(m).\ OI^2+\Sigma\ (\frac{1}{2}\ m.\ IP^2)$, which proves the proposition.

Ex. A body of mass M hanging vertically, draws another body of mass m along a horizontal plane, by means of a string passing over a smooth pulley. If v denote the speed at any given instant, and k the coefficient of friction on the plane, find the motion of the centre of mass, the masses of the string and pulley being insignificant.

At the given instant the centre of mass has a horizontal velocity $mv\div(M+m)$, and a vertical velocity $Mv\div(M+m)$ downwards; therefore its total velocity is $\sqrt{(M^2+m^2)}v\div(M+m)$, in a direction which makes an angle with the horizon, whose tangent is the ratio M/m.

Let T denote the tension of the string; then the acceleration of m is $(T-kmg)\div m$, and that of M is $(Mg-T)\div M$. These are numerically equal, therefore T is equal to $Mm(1+k)g\div(M+m)$, and the acceleration of either body is $(M-km)g\div(M+m)$. The centre of mass has therefore a horizontal acceleration $m\ (M-km)g\div(M+m)^2$, a vertical acceleration $M(M-km)g\div(M+m)^2$, and a total acceleration $(M-km)\sqrt{(M^2+m^2)}g\div(M+m)^2$ in a direction which makes an angle $-\tan^{-1}(M/m)$ with the horizon. Hence the acceleration is constant, and its

direction is the same as that of the velocity at the given instant, and therefore the centre of mass moves in a straight line inclined at an angle $=\tan^{-1}(M/m)$ to the horizon with uniformly accelerated motion.

Examination XX.

1. Define the centre of weight, and prove propositions 1, 2, 3 of art. 171.

2. How may the c. of w. of any irregularly shaped body be experimentally determined?

3. What are the different kinds of equilibrium? Illustrate these by bodies supported 1) at a point, 2) on a plane, 3) on a surface of double curvature.

4. When a body is placed on a plane, state and prove the condition of equilibrium.

5. Define the centre of mass of a body. When does it coincide with the centre of weight? Define centroid; what other names may be used for this term?

6. Find the centroids of 1) a triangular plate, 2) a quadrilateral plate, 3) a cuboid, 4) a triangular pyramid, 5) a right circular cone, 6) a spheroid.

7. Determine algebraical expressions for the position, velocity, and acceleration of the c. of m. of any material system.

8. Prove the corollary to art. 175.

9. What is meant by saying that the motion of a body is represented by the motion of its c. of m.? Enunciate and prove the corresponding proposition for the kinetic energy of a material system.

10. What do we know about the c. of m. of the solar system, and of that of the whole universe?

Exercise XX.

1. Three men support a heavy triangular board, of uniform density and thickness, at its corners; shew that all three exert the same force. Would this be the case if they supported it at the middle points of the sides?

2. Prove that a rhomboidal lamina of uniform density, when placed on a horizontal plane, will rest in equilibrium on any one of its sides, if its plane be vertical.

3. A round table stands on three legs placed on the circumference at equal distances; shew that a body, whose weight is not greater than that of the table, may be placed on any part of it without upsetting it.

4. A uniform rod 1 metre long, and 500 grams mass, is supported horizontally by means of a finger below the rod, 5 cm. from one end, and the thumb at the end over the rod; find the pressures on the finger and thumb.

5. Two strings have each one of their ends fixed to a peg, and the others to the ends of a uniform rod; when the rod is hanging in equilibrium, shew that the tensions of the strings are proportional to their lengths.

6. A heavy bar, of uniform section and density, 3 metres long, is to be carried by two men, one of whom is half as strong again as the other; if the weaker man supports the bar at one end, where should the stronger man support it?

7. The base of a solid right circular cone is in contact with a plane which can be gradually inclined; find the ratio of the altitude of the cone to the radius of the base, in order that the cone may be just on the point of toppling over as it begins to slide down.

8. A uniform triangular plate is suspended from a point by strings attached to its angular points; shew that the tensions of the strings are proportional to their lengths.

9. The lower end of a rigid ladder is fixed, whilst the upper rests against a vertical wall; compare the heights a man and boy can ascend respectively, so that the pressure against the wall may be the same.

10. It is observed that a rod AB, 12 feet long, will balance at a point 2 feet from the end A; but when a body of 100 lbs. is suspended at the end B, the rod balances at a point 2 feet from that end; find the mass of the rod.

11. In Ex. XIII, 6 and 10, find the speeds of the centres of mass after impact and bursting respectively.

12. Find the motions of the centres of mass in the systems described in Ex. VII, 5 and 15.

13. Three particles of equal mass are moving along the sides of a triangle taken in order, with speeds proportional to the sides along which they move respectively; find the velocity of their centre of mass.

14. Find the amount of work required to dig a cylindrical well to a depth of 20 feet, the diameter being 4 feet, and the density of the material raised 2·3.

15. A shaft 100 feet deep is full of water; find the depth of the surface of the water when $\frac{1}{4}$, and also $\frac{1}{2}$ of the work required to empty the shaft has been done.

16. Find the centroid of a thin shell in the shape of a right circular cone.

ANSWERS.

1. Yes. 4. 5 and $4\frac{1}{2}$ kilogrs.-wt. 6. 50 cm. from the other end. 7. $4 : k$. 9. Inversely as their weights. 10. 25 lbs. 11. 2250; 50.

12. Uniform vertical acc. of $\frac{1}{25}g$; uniform acc. of 13·9 at $-\tan^{-1}\frac{3}{2}$ to the horizon. 13. 0.

14. 360,705 ft.-lbs. 15. 50, 70·5. 16. On the axis, at two-thirds of the axis from the vertex.

CHAPTER XXI.

Simple Machines.

187. This chapter might be called an introduction to mechanics (art. 61). Machines are used for various purposes: for transmitting force or motion, as in many uses of flexible cords and straps, rigid rods, or toothed wheels; for changing the direction of force or motion, as when a stretched rope is passed around a pin or smooth pulley, or in the use of bevelled toothed wheels; in measuring force, as by a balance or dynamometer; but the use principally aimed at, in the larger number of machines, is to enable man to balance, or do work against great forces by the application of small forces. This is the use, for example, of a crowbar to balance the weight of or to move a heavy beam, or of a screw to apply great pressure, as in an ordinary book-binder's press. This may be called appropriately the *dynamical advantage* of the machine.

If W denotes the resistance which is balanced or worked against by the aid of the machine, and P the force applied to balance or work against the resistance, the ratio W/P measures the dynamical advantage.

In the present chapter the dynamical advantages of a few of the simplest machines will be calculated. More complicated machines will be found to be combinations of such simple machines, and if the dynamical advantages of the latter are known, it needs but simple multiplication to find those of the former.

188. When there is no appreciable friction called into play in the use of a machine, the dynamical advantage arises entirely from the combination of parts or the mechanism of the machine, and is then called the *mechanical advantage* of the machine. Thus in sliding a heavy body

up an inclined plane, which may be considered a simple machine, the mechanical advantage is $(\cos b)/(\sin a)$, art. 153 cor 1, and this becomes cosec a, when P acts along the plane, and cotan a, when P acts horizontally.

When friction is called into play, and the machine is used to aid in merely balancing the resistance W, then friction can always be taken advantage of in favour of the balancing force P. When friction is fully taken advantage of, so that the resistance W is just kept in check, or motion is just about to take place against P, the dynamical advantage may then be called the *static advantage* of the machine. It is the greatest advantage the machine can offer in balancing any resistance. Thus, in the inclined plane, the static advantage is $\cos(b+f)/\sin(a-f)$, art 153. and this becomes $1/(\sin a - k \cos a)$ when P acts along the plane, and $\cot(a-f)$ when P acts horizontally.

When it is desired to do work, *i.e.* move against the resistance W, with the aid of the machine, the friction called into play always acts in opposition to the moving force P. In this case the dynamical advantage may be called the *kinetic advantage* of the machine. It is evidently always less than the mechanical advantage. Its value, like that of the static advantage, depends upon the friction called into play as well as on the mechanism of the machine. In the inclined plane the kinetic advantage is $\cos(b-f) \div \sin(a+f)$. art. 153, which becomes $\cot(a+f)$ when P acts horizontally, and $1/(\sin a + k \cos a)$ when P acts along the plane.

189. *Less work is never done in overcoming any resistance through a given distance with the aid of a machine, than would be done in overcoming the resistance through the same distance directly, i.e., without the machine.*

Thus, although a man may be able to roll a heavy body which he could not lift up an inclined plane, yet the amount of work he does is not less than the work which would be done in raising the heavy body vertically through

the height of the plane. In reality more work is done with the aid of the machine, for some work must always be done against friction, and this work is entirely lost. It is a case of the dissipation of energy (art. 122).

If there were not any friction, the work done with the aid of a machine by the moving force would be the exact equivalent of the work done against the resistance.

This is an immediate result of the great law of the conservation of energy (art. 120), and is generally known as the *principle of work*. Thus, in sliding a body up an inclined plane, $P \cos b = W \sin a$, (art. 153, cor. 1), therefore $Pl \cos b = Wl \sin a$; but $l \cos b$ is the distance through which P works in its own direction in raising the body from the bottom to the top of the plane, l denoting the length of the plane, and $l \sin a$ is the vertical distance through which W is overcome; hence the work done by P is equal to the work done against W. If P act along the plane, $P \times$ length of plane $= W \times$ height, (art. 153, cor. 3).

190. It is evident that from the principle of work the mechanical advantage of a machine can be determined with great facility by studying the kinematics of the machine. It is not always vital energy which is used as the motive power in machines. In a steam-engine *e.g.* it is the potential energy of atomic separation of coal and the oxygen of the air, which is transformed into mechanical work, and if we do not consume animal energy in doing work with a steam-engine, we exhaust an equivalent amount of our store of potential energy in the form of coal.

The Lever and Fulcrum.

191. A lever is a rigid rod moveable in one plane about an axis called the fulcrum. The condition of equilibrium can be deduced from art. 168. When, as is generally the case, friction may be neglected, and the weight of the lever itself is balanced at the fulcrum or may be neglected, the condition can be stated thus: The moment of the balanc-

ing or moving force about the fulcrum must be equal and unlike to the moment of the resistance.

Hence the mechanical advantage of a lever is measured by the ratio of the arm of the moving or balancing force about the fulcrum to the arm of the resistance. This can be deduced easily from the principle of work by making a small displacement about the fulcrum. To find the pressure on the fulcrum, apply art. 143 or 159.

192. Levers are sometimes divided into 3 classes: 1) those in which the fulcrum lies between the places of application of the balancing or moving force and resistance, as a crow-bar, a claw-hammer, a common balance, or common scissors; 2) those in which the place of application of the resistance lies between the fulcrum and place of application of the balancing or moving force, as in a wheel-barrow, an oar, nut-crackers, or cork-squeezers; 3) those in which the place of application of the moving force lies between the fulcrum and the place of application of the resistance, as in many parts of the animal frame-work, such as the forearm, in the shells of bivalve molluscs, and some forms of shears. In the first and second classes the mechanical advantage will generally be greater than unity, and in the third less than unity. The object sought in the third class of levers is therefore not the acquisition of mechanical power, but the production of motion over a considerable range by means of motion through a small distance.

Many very powerful combinations of levers are used in the mechanical arts, such *e.g.* as may be seen in the hand printing presses, and the large paper cutting machines used by printers and bookbinders.

Wheel and Axle.

193. A wheel and axle consists of two cylinders capable of rotating about a common axis. The larger cylinder is called the wheel, and the smaller one the axle. To the

latter a resistance to be worked against is applied by means of a rope in tension or otherwise, to the former a moving force to overcome this resistance. Familiar examples of such machines are found at wells to draw water up in buckets, and in the windlass and capstan which are commonly used on board ship, the former to raise merchandise from the hold, the latter to weigh anchor. In these the wheel generally consists of spokes, at the extremities of which the moving force is applied. The axis of the capstan is vertical, and the applied forces horizontal.

The mechanical advantage can be easily deduced from art. 168, or from the principle of work, thus: Let R and r denote the radii of the wheel and axle respectively; if the machine be rotated through any angle i, the work done by the moving force P is PRi, and that done against the resistance W is Wri; since these must be equal, $W/P = R/r$. The pressure on the axis of rotation will be the resultant of P and W, and the weight of the machine.

Toothed Wheels.

194. Toothed wheels are principally used to communicate motion from one wheel to another, as *e.g.* in clockwork. In cranes and other machines, however, mechanical advantage is sought by their aid. We shall illustrate their use in this respect by finding the mechanical advantage of a machine consisting of one wheel and axle driving another wheel and axle, the axle of the first and the wheel of the second having teeth which work into one another. Let P denote the force acting on the first wheel, which balances, without friction being called into play, a resistance W acting on the second axle. Let Q denote the mutual pressure between the teeth in contact; this will be normal to the surfaces in contact. Denote by R and l the arms of P and Q about the first axis, and by l' and r' the arms of Q and W about the second axis. Then (art. 168),

$$PR = Ql,\ Ql' = Wr',\ \therefore\ W/P = Rl'/lr'.$$

If r and R' denote the radii of the first axle and the second wheel, and the teeth be small compared with these radii, $l/l' = r/R'$, and therefore $W/P = RR'/rr'$, a result easily deduced from the principle of work, thus: Let the first wheel and axle be rotated through any angle i, then the second will be rotated through an angle $i.\ (r/R')$, and therefore $PRi = Wr'i.\ (r/R')$, or $W/P = RR'/rr'$.

Endless bands are much used in machinery for the same purposes as toothed wheels, when it is inconvenient to bring the wheels close together, friction preventing the bands from slipping.

Pulley and Rope.

195. A pulley is a grooved wheel which rotates about an axis, fixed generally in a framework called a block. A stretched rope or cord passes around the wheel within the groove. The pulley may be used 1) merely to change the direction of the rope, so as to apply a force in a convenient direction, or 2) to get mechanical advantage. In the first case the pulley is fixed, in the second it is moveable.

To understand more clearly what follows, the student may imagine the part of the cord, which is in immediate contact with the pulley at any instant, as forming part of the pulley to which the forces are applied. Neglecting friction, the tension of the rope wound round the pulley is the same at both sides (art. 141). To balance these tensions, there must be another force acting on the pulley equal to $2\ T \cos i$, where T denotes the tension of the rope, and $2\ i$ the inclination of its two branches as it leaves the pulley. If 1) the pulley be fixed, $2\ T \cos i$ denotes the tension of the beam which holds the pulley, necessitated by the tension of the rope; if we add to this the weight of the pulley and rope, we get the total tension of the beam. If 2) the pulley be moveable, $2\ T \cos i$ denotes the resistance which the tensions of the rope balance, and this resistance includes the weight of the pulley and rope, which, however, are often neglected.

Cor. If the two branches of the rope be parallel in a moveable pulley, and W denote the resistance, $W = 2\ T$, a result easily arrived at from the principle of work.

196. Pulleys in various combinations are commonly used in practice to lift bodies against their weights. The student will find the above results sufficient to enable him to determine the mechanical advantage, as well as the tension of the supporting beam, in any combination whatsoever. In any system of pulleys the kinetic advantage is considerably less than the mechanical on account of the friction called into play, arising principally from the rigidity or imperfect flexibility of the ropes.

Double Axle and Pulley.

197. A combination, in which the mechanical advantage can be made as great as required with great facility, is known as the *differential axle* or *Chinese wheel*. It consists of two cylinders having a common axis; round the larger cylinder a rope is wound a few times, then passed under a moveable pulley, and thereafter round the smaller cylinder. The direction of coiling the rope on the latter is opposite to that on the former, so that as the rope is wound on to the larger cylinder it is wound off the smaller. The moving force is generally applied by means of a winch which rotates on the same axis as the double axle, and the resistance acts on the block of the pulley. Let a denote the arm of the moving force applied to the winch, b and c the radii of the larger and smaller cylinders respectively. If the two lines of rope passing round the pulley be parallel, then for one complete turn we get by the principle of work, $P.2\pi a = W.\pi(b-c)$, and therefore $W/P = 2a/(b-c)$.

Hence, by making the difference between b and c small enough, the mechanical advantage can be made as great as required without sacrificing the strength of the machine or making it unduly bulky.

Screw and Nut.

198. A screw may be described as a right circular cylindrical bolt, on the surface of which runs a uniform projecting thread, which makes a constant angle with the base. The nut of a screw is a hollow cylinder in which is cut a spiral groove, the exact counterpart of the thread of the screw.

When the screw enters its nut, it is evident that they can only move relatively to one another by one of them rotating, and then there is sliding motion parallel to the axis of the cylinder, proportional in amount to the angle of rotation. Either the screw or nut can both slide and rotate, or one can rotate and the other slide. Generally the nut is fixed, as in a book-press, or the screw only rotates and the nut slides, as in a dividing engine.

The distance, measured parallel to the axis, between two adjacent coils, is called the *pitch* of the screw. It is evident that for every complete rotation the sliding motion is equal to the pitch of the screw. The inclination of the thread to the base of the cylinder is called the *angle* of the screw.

The principal uses of a screw and nut are 1) to measure small distances, as in a dividing engine or micrometer; and 2) to exert great pressure in the direction of the axis, as in a book-press. For the first of these uses, the head of the screw is provided with a carefully divided circle to enable the experimenter to measure any small rotation; for the second, the moving force is generally applied to a rigid bar fixed into the head of the screw, and acts at right angles to the axis. If a denote the arm of the moving force and p the pitch of the screw, it is evident that the mechanical advantage is measured by $2\pi a/p$.

Screw and Toothed Wheel.

199. In this machine a toothed wheel takes the place of a screw's nut. The thread of a short screw fits into the

spaces between the teeth of a wheel (or, the teeth of the wheel fit into the groove between the coils of the screw). The axis of the screw is fixed, and so long as the screw is turned, the wheel is made to rotate about its own axis. Hence the machine is generally called the *endless screw*. The wheel generally forms part of a wheel and axle, and to the axle the resistance is applied. The moving force is applied by means of a crank or winch fitted on to the axis of the screw. The mechanical advantage is evidently measured by the product of the numbers, which measure the mechanical advantages of the screw, and wheel and axle respectively.

The Wedge.

200. The wedge may be described as a rigid right triangular prism having two of its faces inclined generally at a very acute angle. The line in which those faces meet is called the *edge* of the wedge, and their inclination the *angle* of the wedge. The face opposite the edge is called the *head* of the wedge. The wedge is practically used to separate two bodies, as in lifting a heavy body through a small distance against its weight, or to divide a body into two parts against molecular force, as in splitting wood. Friction plays an important part in the practical use of the wedge. The moving force is frequently impulsive, as when the wedge is driven by the sharp blows of a hammer. Axes, knives, and chisels are different forms of the wedge, which may be considered a double inclined plane, and as the angle between the two faces is very small the mechanical advantage is very great.

In the above machines there are in reality only three primary principles, viz., the principles of the inclined plane, the lever, and the moveable pulley. The wheel and axle and toothed wheels are just special levers, whilst the screw and nut and the wedge are special forms of the inclined plane.

Examination XXI.

1. Define the mechanical, static, and kinetic advantages of a machine, and find those of an inclined plane, when a body is slid up the plane.

2. Enunciate the principle of work and apply it to find the mechanical advantages of the lever. Chinese wheel, and endless screw.

3. Give examples of the different classes of levers, and state the object of each class.

4. What is the effect of changing the direction of P or of W in a wheel and axle? Is the mechanical advantage thereby changed?

5. In the *first system of pulleys* each pulley is supported by a separate rope, one end of which is fixed to the beam, and the other to the block of the pulley above. If the lines of rope be all parallel, and the free end of the rope of the highest moveable pulley pass over a fixed pulley, find the mechanical advantage, 1) when the weights may be neglected, 2) when the weight of each pulley is w. Find also the tension of the supporting beam.

6. In the *second system of pulleys* there are two blocks, one fixed and the other moveable; each block contains a number of pulleys, and the same rope passes round all the pulleys, the lines of rope being parallel or very nearly so; find the mechanical advantage, 1) when the weight of the lower block and pulleys may be neglected, 2) when the weight of the lower block and pulleys is w. Find also the tension of the beam supporting the upper block.

7. The *third system of pulleys* is just the first system reversed, the beam and resistance changing places; find in it the mechanical advantage and tension of the beam, 1) when the weights of the pulleys are insignificant, 2) when the weight of each pulley is w.

8. When will the static and kinetic advantages of an inclined plane be maxima for given values of a and k?

Exercise XXI.

1. Prove that the *efficiency* of a machine (art. 122) is measured by the ratio of the kinetic advantage to the mechanical advantage, and hence find the efficiency of an inclined plane.

2. Shew how to graduate a common steelyard. What change is produced on the graduations by increasing 1) the moveable counterpoise, 2) the density of the rod.

3. If a counterpoise of one pound be used in a steelyard which was graduated for a counterpoise of one kilogram, shew that the merchant will defraud himself, or defraud his customers, or deal justly, according as the centre of weight of the steelyard is in the longer arm, in the shorter arm, or just below the point of suspension.

4. A lever of insignificant weight is a metre long; a body of 10 kilograms is supported by two strings, 6 and 8 decimetres long respectively, attached to its extremities; if the lever be in equilibrium when horizontal, shew that the fulcrum divides it into two parts in the ratio of 9 to 16. What is the pressure on the fulcrum?

5. In the Danish steelyard the beam is heavy at one end and the fulcrum moveable; the masses to be measured are suspended at the light end; shew that the distances of the graduations from the light end form an harmonical progression.

6. Under what condition may there be no mechanical advantage in the first system of pulleys, w being the weight of each pulley? Find the force required just to balance the weights of the pulleys.

7 The mass of a uniform straight lever, which turns about a fulcrum at one end, is 6 lbs.; in what direction must a force of 5 lbs.-wt. be applied at the other end, so as to keep the lever at rest in a horizontal position ? What will be the pressure of the fulcrum ?

8. Sixteen sailors, each exerting a force of 30 lbs.-wt., push a capstan, each with a length of lever equal to 8 ft.: calculate the weight they are capable of sustaining, the radius of the cylinder of the capstan being 1 ft. 4 in.

9. A body of 100 kilograms is kept from sliding down an inclined plane of inclination $\frac{1}{6}\pi$, by a string in tension parallel to the plane; the string passes round an axle of diameter 3 decimetres; find what body must hang from a wheel, of diameter 1 metre, having a common axis with the axle, in order to keep the 100 kilograms at rest without friction being called into play.

10. One body is suspended from a single moveable pulley, and is supported by another body hanging freely over a fixed pulley, the three lines of rope being parallel: prove that, whatever be the vertical height of each body, the height of their centre of weight is constant.

11. A man stands in a scale attached to a moveable pulley; the free end of a rope passes over a fixed pulley; find with what force the man must hold the free end, in order to support himself, the lines of rope being parallel.

12. One tonne is to be raised by means of a third system of 6 pulleys; if the mass of each pulley be a kilogram, find the mechanical advantage, the tension of each rope in kilograms-weight, and the tension of the beam.

Answers.

1. $\{\text{Cos}\,(b-f)\sin a\}\div\{\cos b\sin(a+f)\}$. 2. The zero is moved 1) nearer to, 2) further from the fulcrum.
4. 10 kilograms-wt. 6. $W=w$; $w-w/2^n$.
7. $\sin^{-1}\frac{3}{5}$ to the horizontal; 5 lbs.-wt.
8. 2880 lbs.-wt. 9. 15 kilogrs. 11. One-third of the weight of the man, scale, and pulley.
12. 66·8; 15, 31, 63, 127, 254, 510; 1021.

Miscellaneous Examples.

1. It requires a grams-wt. to sink Nicholson's hydrometer by itself to the mark on the stem, b grams-wt. when a piece of amber is placed in the upper pan, and c grams-wt. when the amber is placed in the lower pan; find the s.w. of amber.

2. A man whose mass is 68 kilograms can just float in fresh water; find the greatest quantity of gold (s.w. 19·3) he could keep from sinking, when floating in the sea (s.w. 1·027).

3. If in Ex. XIII, 13, the balls be painted alike so that the frictional resistance f between the water and each ball be the same, will the resistance increase or diminish the tension of the cord? By how much?

4. Two bodies of 4 and 5 kilograms together pull one of 6 kilograms over a smooth peg by means of a connecting string; after descending through 10 metres, the 5 kilograms mass is detached without interrupting the motion; find through what distance, and for what time, the remaining 4 kilograms will continue to descend.

5. To a person travelling eastwards with a speed of 4 miles per hour the wind appears to be north; on doubling his speed, it appears to be N.E.; find the velocity of the wind.

6. A body of 6 lbs. hanging vertically is connected by an inextensible string with a body of 4 lbs. which is drawn up a plane inclined to the horizontal at $\frac{1}{3}\pi$ radians; find the motion of the c. of m., $k=\frac{1}{4}$.

7. The hole in a boiler for the safety valve is a circle of $\frac{5}{8}$ in. diameter. The centre of the valve is $1\frac{3}{8}$ in. from the fulcrum of a lever which keeps it closed; find where a weight of 7 lbs. must be hung to the lever so that the valve may not rise till the pressure of the steam is 3 atmospheres. Take $\pi=\frac{22}{7}$ and 1 atmosphere=14·7 lbs.-wt. per sq. in.

8. In a Bramah press the diameters of the pistons are 2 and $\frac{1}{2}$ inch. The smaller piston is also the piston of a pump which supplies the liquid to the press. The arms of a lever which moves this piston are $2\frac{1}{4}$ and $11\frac{1}{4}$ inches: find the mechanical advantage of the machine.

9. A ship is sailing eastwards, and it is known that the wind is N.W.: the apparent direction of the wind, as shown by a vane on the mast head, is N. N.E.: shew that the speed of the ship is the same as that of the wind.

10. A ship sailing eastwards with a speed of 15 miles per hour passes a light-house at noon: a second ship sailing northwards with the same speed passes the light-house at 1.30 p.m. When were the ships nearest to one another, and what was their distance apart then?

11. *ABCD* is a parallelogram; forces represented in magnitude and line of action by *AB*, *BC*, and *CD* act upon a body: find the resultant.

12. A man's mass is 140 lbs., and he supports an English ton-weight (2,240 lbs.) by means of 4 moveable pulleys fixed as in the first system: find his pressure on the floor on which he stands, 1) when he pulls the free end of the rope upwards, 2) when the rope passes over a fixed pulley, and he pulls downwards.

13. The capacity of the receiver of an air-pump is 20 times that of the barrel; a piece of bladder is placed over a hole in the top of it: the bladder is able to bear a pressure of 3 lbs.-wt. per sq. in.; how many strokes of the pump will burst the bladder?

14. A body floats in water with $\frac{1}{4}$ of its volume above the surface; the whole is put under the receiver of an air-pump, and the air extracted: find the alteration in the volume immersed.

15. How much cork (s. w. $\frac{1}{4}$) is required to float a man of 152 lbs. in sea-water, his mean s.w. being 1·1?

16. Find the lines of quickest descent between a point and a line, and shew that they are at right angles to one another.

17. An endless cord hangs over two smooth pegs in the same horizontal line, and a heavy body is supported on each festoon; if the one body be twice as heavy as the other, shew that the angle between the lines of the upper festoon must be greater than $\frac{2}{3}\pi$ and less than π.

18. A body floats in a liquid two-thirds immersed, and it requires a pressure equivalent to two lbs.-wt. just to immerse it totally; what is the mass of the body?

19. Oxygen at 0° and 76 cm. pressure has density 1·1056 with respect to air; find its density at 100° and 70 cm., 1) with respect to air at 0° and 76 cm., 2) with respect to air at 100° and 70 cm.

20. The mass of a specific gravity bottle is 20·5 when empty, 70·5 when filled with water, 63 when filled with turpentine; when 10 grams of salt are put into it, and it is thereafter filled up with turpentine, the mass is 69·6; find the s.w. of the turpentine, and of the salt to an approximation of the first degree.

21. A sunken vessel, whose bulk is half a megalitre and mass 10^6 kilograms, is to be raised by attaching water-tight barrels to it. If the mass of each barrel be 30 kilograms, and the volume a kilolitre, find how many will be required.

22. Find the height of the water barometer under the mean atmospheric pressure, when the temperature is 15° C, the s.w. of water at 15° being 0·999125 according to Despretz.

23. One end of a string is fastened to a body of 10 kilograms; the string passes over a fixed pulley, then under a movable pulley, and has its other end attached to a fixed hook; a body of $7\frac{3}{4}$ kilograms is attached to the movable pulley, whose mass is 250 grams; if the three parts of the

string be parallel, and friction and the masses of the string and pulleys may be neglected, find the accelerations of the bodies and the tension of the string.

24. A body of 100 kilograms pulls by its weight 200 kilograms along a rough horizontal plane; if the coefficient of friction be 0·2, find the speed after moving through a hectometre, and the acceleration of the c. of m.

25. A stream is a feet broad, b feet deep, and flows at the rate of c feet per hour; there is a fall of d feet; the water turns a machine of which the efficiency is e; it requires f foot-pounds per minute for 1 hour to grind a bushel of corn; determine how much corn the machine will grind in 1 hour.

26. Find what must be the area of a cake of ice 18 inches thick, sufficient to bear the aggregate weight of three school boys whose aggregate mass is 280 lbs.; 1) in fresh water, 2) in sea-water.

27. Three bodies P, Q, R, of masses 30, 15, 10 kilograms respectively, are connected by strings AB and BC, whose lengths are 5 m. and 70 cm. Q, R, BC, and half of AB lie on the edge of a table vertically under a peg, over which the other half of AB is placed holding P. If P be now allowed to fall freely, find the motions of P, Q, and R, the tensions of the strings after both become stretched, and the measures of the impulsive tensions which set Q and R in motion. Friction and the masses of the strings may be neglected, and $g=980$.

28. A string which passes over two pegs in a horizontal line supports a heavy ring from falling; prove that the string cannot be drawn so tight as to be horizontal.

29. A, B, C, and D are any four points whatsoever; find a point O such that forces represented by OA, OB, OC, and OD are in equilibrium. Hence shew from dynamical considerations that the lines joining the middle points of AB and CD, of AC and BD, and of AD and BC bisect one another.

30. A cubical box is all but $1/n$th part filled with water, and is placed on a rough rectangular board so as to have the edges of the base parallel to those of the rectangle; determine in what order spilling, sliding, and toppling over will take place, when the board is gradually inclined to the horizon about an edge.

31. Find the least volume of a balloon filled with hydrogen that it may rise from the earth when the mass of the solid parts of the balloon and the contents of the car is altogether 250 kilograms.

32. The height of mount Fuji in Japan was found by means of an omnimeter to be 12365 feet; the reduced reading of the barometer on the summit was found to be 48 cm. when the temperature was 0°; shew that the height of the atmosphere is at least 7·3 miles high.

ANSWERS.

1. $(a-b)/(c-b)$. 2. 1939·2. 3. Increase by $\frac{7}{8}f$ very nearly. 4. 10 m., 10·2 sec. 5. $4\sqrt{2}$ miles per hr., N.W. 6. $(11-4\sqrt{3})\sqrt{(13-6\sqrt{3})g}$ $\div 100$ in direction $-\tan^{-1}(3-\sqrt{3})$ to the horizon.
7. 2·66 inches from the fulcrum. 8. 80.
10. 12.45 p.m.; 15·9 miles. 12. 1) 280 lbs.-wt., 2) 0.
13. 4·7. 14. An increase of $3{\cdot}233/10^3$ of the volume.
15. 3·25. 18. 4. 19. 0·7453, 1·1056.
20. 0·85, 2·5. 21. 516. 22. 1016·9.
23. $\frac{1}{2}g$, $\frac{1}{4}g$, 5 kilogrs.-wt. 24. 1980·4; $\frac{1}{15}g\sqrt{5}$ at $-\tan^{-1}\frac{1}{2}$ to the horizontal. 25. $62{\cdot}4\,abcde/60f$.
26. 37·4, 28. 27. Q starts with speed 1400/3, R with 420; $27\frac{3}{11}$ and $10\frac{10}{11}$ kilograms-wt.; Q, AB 7 megagramtachs; R, AB 2·8 and BC 4·2 megagramtachs.
29. The middle point of EF, E and F being the middle points of AB and CD. 30. It spills when $\tan\alpha = 2/n$ or $n/(2n-2)$, slides when $\alpha = f$, and topples over when $\alpha = \frac{1}{4}\pi$. 31. 207,710 litres.